A Matrix Completion Approach to Policy Evaluation: Evaluating the Impact of the VCT Scheme on Investment in the U.K.

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Abstract

This study estimates the causal effect of the Venture Capital Trust (VCT) scheme on investment (change in total-assets formation) for investees in the U.K. To that end, we hand-collect data on all firms (investees) that received VCT funding, from inception of the scheme till 2018. Thereafter, we adapt an unsupervised machine learning algorithm called matrix completion to estimate causal effects in settings where some investee-years are exposed to a binary treatment (VCT funding) and the goal is to estimate counterfactual outcomes for the investee-years combinations. In tandem with the hand-collected data, we use the matrix completion algorithm to estimate the causal effect of the Venture Capital Trust (VCT) scheme on the investment of investees. The estimand is the Average Treatment Effect on the Treated (ATT). We find that the VCT scheme caused a 41% increase in the investment of investees; the ATT is 41%. We also document novel insights regarding the relationship between changes to the U.K. government VCT policy, VCT fundraising and the aggregate investment of VCTs. Finally, we show that the matrix completion estimator outperforms an unconfoundedness-based estimator and alleviates the potential selection bias issue inherent in a causal study like this study.

Keywords: Venture Capital Trust, Causal Effects, Hand-Collected Data, Low-Rank Matrix Estimation, Matrix Factorisation

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1 Introduction

The Venture Capital Trust (VCT) scheme, introduced in 1995, is one of three tax-based venture capital schemes, the others being the Enterprise Investment Scheme (EIS) and the Seed Enterprise Investment Scheme (SEIS). The VCT scheme is a U.K. government policy response to a perceived breakdown in financial markets and their ability to provide risky capital to risky but promising U.K. SMEs. It is designed to encourage investors to invest (indirectly) in British, unquoted, smaller, and higher risk firms - with a need for start-up, early stage or expansion capital - by investing through subscription to a VCT's shares. VCTs are U.K. publicly-quoted and closed-ended funds, and the U.K. government encourages investment in these financial intermediaries by offering tax-rebates to investors. The VCT scheme has broad base appeal, not least because at the minimum it increases the supply of finance, thus creating value for the U.K. economy, but also we know from Haltiwanger, Jarmin, and Miranda (2010) that SMEs make outsized contributions to net employment growth in an economy. Indeed, Gonzalez-Uribe and Paravisini (2019) show that the SEIS scheme caused a 10% decrease in the cost of outside equity for young firms and a 1.6% increase in the investment of young firms. They also find that conditional on the issuance of new equity under the SEIS scheme, young firms increase their investment by 8 times the equity issuance.

We find that the VCT scheme has had a significant effect on the growth of small, young firms in the U.K. It caused a 41% increase in the total-assets formation (investment) of investees between 2003-2018. Total-assets formation or investment is the change in total-assets. Based on the stated aim of the VCT scheme - which as noted earlier is to stimulate the growth of young risky firms in the U.K. - we believe our choice of total-assets formation (investment) as the measure of the efficacy of the VCT scheme, is the most relevant measure of entrepreneurial firm growth. However, data limitations prevent us from utilising additional measures such as patents and macro aggregates i.e. contribution to employment growth, contribution to GDP, to name a few.

The main objective here is to develop a selection-bias-alleviating algorithm to estimate then quantify the causal effect of the VCT scheme on total-assets formation of investees in the U.K. We adopt the Athey et al. (2018) machine learning Matrix Completion framework for estimating causal effects in a setting where some firm-years are subject to a binary shock. Specifically, the Matrix Completion framework helps to impute the counterfactual ("missing") total-assets of investees, which then allows us to estimate the average causal effect of the VCT scheme on the total-assets formation (investment) of investees (The Average Treatment Effect on the Treated; ATT). The Matrix Completion framework helps alleviate the potential selection bias and identification issues in this study. In our causal effect setting, selection bias could be due to unobserved differences across investees and non-investees control group. The selection bias could thus drive our finding of a positive causal effect, as opposed to the VCT scheme, because investees are different from the non-investees control group. Let us consider the popularly used Difference-in-Differences (DID) framework. The VCT scheme or the VCT funding of an investee becomes endogenous when selection bias causes VCTs to invest in investees that are superior to non-investees along several dimensions that are unobserved in the data. Investees with superior unobserved features, as reflected in the error term in the DID, received VCT funding. Thus, the error term is positively correlated with VCT funding (investees total-assets) and the estimated causal effect is biased upwards relative to the VCT scheme's actual causal effect on total-assets formation. This selection bias issue is even more prevalent among small, young and risky firms - the focus of the VCT scheme. These investees have very limited operating and financial histories. The Companies House reporting standards are also less strict for these small, young and risky firms - relative to bigger, older and more established firms. Additionally, and as evinced in Gompers, Gornall, Kaplan and Strebulaev (2020), VCTs emphasise numerous unobservable factors such as the management team, uniqueness of product, market and industry competition, when they screen potential investees. With the DID approach, a classic solution to the selection bias issue is the parallel trends assumption, which in this study would be that the total-assets formation (investment) of investees and non-investees control group would have evolved identically, but for the VCT scheme. This assumption is then tested by comparing the total-assets formation of both investees and non-investees control group prior to receiving VCT funding. Unfortunately, this test does not resolve the selection-along-unobservable-characteristics issue. Not to mention, and as detailed in Athey et al. (2018), researchers have to make an ex-ante choice between exploiting cross-sectional or time-series correlation patterns or a combination of both, to create a non-investees control group. The Matrix Completion approach alleviates the selection-along-unobservable-characteristics issue and the ex-ante choice issue by allowing the data to drive what correlation patterns are exploited in the data, to construct the counterfactual outcome (total-assets). For instance, the Matrix Completion framework exploits patterns in the total-assets observations of non-investees (control group), and crucially, in the pre-VCT-funding total-assets observations of investees (treated firms), which implies that the counterfactual total-assets are constructed from a hybrid of data patterns as opposed to data patterns extracted from any one control group.

A crucial first step in this study is the hand-collection of data on all investees in the U.K. (both former and current). To our knowledge, our hand-collected data on investees is the most comprehensive VCT data available.[1] Our hand-collection efforts also allows us implement the secondary objective of this study, which is to employ hand-collected information from VCT annual reports to conduct analysis on how changes to the governmental regulations guiding VCT activities shaped VCT fundraising and the aggregate investment patterns of investees. The Athey et al. (2018) Matrix Completion framework that we adopt differs from but combines the unconfoundedness and synthetic control frameworks. Given an observed matrix of outcomes for units - which could include both treated and untreated units - they assume that the data for treated units during treatment periods is missing. The task is to impute the missing entries for treated units in the matrix. The imputed values represent the counterfactual ("missing") outcomes. For this study, it implies imputing the counterfactual total-assets of investees. This approach to imputing counterfactual ("missing") entries in a matrix assumes that the complete matrix has a low-rank, a rank we can implicitly realise by regularisation methods (by adding a penaliser to the objective function), and an approach that has been employed in seminal studies in the matrix completion literature such as Cai, Candes, and Shen (2008), Candes and Recht (2009), Candes et al. (2009), Candes and Plan (2009), Keshavan, Oh, and Montanari (2009). The literature on causal inference has several approaches to the problem of imputing the counterfactual ("missing") outcomes. For instance, Imbens and Rubin (2015) take an unconfoundedness approach to the problem. This approach is akin to imputing the counterfactual ("missing") outcomes for treated units with the observed outcomes for control units - which are units that share similar pre-treatment outcome values with the treated units. Another approach is the synthetic control approach employed in studies such as Doudchenko and Imbens (2016) and Ben-Michael, Feller, and Rothstein (2018). This approach is akin to imputing the counterfactual ("missing") outcomes for treated units with

[1] Popularly used platforms for data on VC Deals have very sparse coverage of VCTs that are unaffiliated with VCs.

weighted average outcomes for control units. Here, the weights are constructed such that the weighted lagged control outcomes are equal to the lagged outcomes for treated units.

Athey et al. (2018) note that whilst the unconfoundedness and synthetic control approaches are similar, they have very salient differences. They especially differ in the data-correlation patterns they exploit to impute the counterfactual ("missing") outcomes. The unconfoundedness approach assumes that the outcomes for the treated and control units follow the same trend in the pre-treatment period. Also, the typical setting in the unconfoundedness approach is one in which the treated units are assumed to be treated all at the same time, in the last period. In contrast, the synthetic control approach assumes that the correlation between outcomes for both control and treated groups are steady over time. Whereas, the typical setting in the synthetic control approach is one in which there are only one or a few treated units, significantly more control units, and a substantial number of pre-treatment periods. Athey et al. (2018) argue that, given a particular setting, both approaches are interchangeable - after some regularisation. Indeed, they show that the unconfoundedness and synthetic control approaches can also be viewed as matrix completion approaches based on matrix factorisation. However they show that the matrix completion approach has a superior performance due in part to its use of regularisation to characterise the estimator, whereas the unconfoundedness and synthetic control approaches impose restrictions on the factors in the matrix factorisation.

We now turn to re-emphasising the importance of venture capital (the wider framework within which the VCT scheme operates) funding for SMEs and by extension the economy. Kaplan and Lerner (2010) document that even though less than 0.25% of U.S. firms receive VC-backing, an estimated one-half of IPOs are VC backed, Metrick and Yasuda (2011) emphasise the positive relationship between VC funding, small firms, and innovation, and Gompers, Gornall, Kaplan and Strebulaev (2020) document: the VC-backed heritage of numerous innovative companies, their effects on the U.S. and global economy, and with the aid of survey data - explore how these VCs make decisions. However, although VCTs are analogous to VCs,[2] the specific importance of VCT funding for SMEs and in turn, the wider economy, is practically unknown in academia.[3]

[2]The main difference between VC and VCT primarily centres around the trust status and specific government regulations guiding VCTs

[3]The bulk of knowledge on VCTs and their importance to macroeconomic considerations is limited to reports commissioned by various bodies such as: governmental agencies, VCTs themselves, and investment companies and

The primary contribution of this paper is the quantification of the causal impact of the VCT scheme on the total-assets formation (investment) of investees in the U.K. We find that the VCT scheme has had a very discernible effect on total-assets formation (investment) in the U.K. It led to a 41% increase in the total-assets formation (investment) of investees between 2003-2018. Finally, our VCT data hand-collection efforts allow us to extract information with which we analyse the relationship between: contemporaneous changes to the VCT regulations, annual VCT fundraising, and the annual aggregate investment patterns of investees. This particularly can serve as a template for regulators to enact effective changes to the VCT regulations. For instance, we note how changes to the age criteria for first-time investees immediately affected the median size of new investees. This can inform regulators on what policies to implement to immediately affect the type of firm that receives VCT funding.

The remainder of this study is organised as follows. In section 2, we describe VCTs and summarise the tax benefits of investing in them. In section 3, we detail the investee data hand-collection process. We also present two separate summary statistics on investees and VCTs - the first is based on our hand-collected data, the second is based on Her Majesty's Revenue and Customs (HMRC) VCT data. Section 4 provides the framework for our estimand, the Average Treatment Effect on the Treated; ATT. In section 5, we present the Matrix Completion estimator. In section 6, we present our main results, the causal effect of the VCT scheme on the investment of investees in the U.K. (ATT). We also analyse how major VCT policy changes impacted VCT fundraising and the aggregate investment patterns of investees. Finally, we present some additional results comparing the performance of our Matrix Completion estimator with a Difference-in-Differences (DID) estimator. In section 7, we summarise and conclude. The appendix contains illustrations of the tax benefits from investing in VCTs, detailed steps on the closed form and numerical solution for our Matrix Completion estimator, and the major VCT policy changes between 1995-2020.

their affiliates.

2 All About VCTs

Before we get into the data hand-collection, analysis and results, it is useful to provide a detailed insight into VCTs and what they are about. VCTs fall under three broad categories: generalist (VCTs that fund firms in various economic sectors), AIM (VCTs that fund firms listed on the AIM market), and specialist (VCTs that fund firms in one or a few sectors e.g. renewable energy infrastructure, technology, or media). A VCT appoints a regulated investment manager who invests and manages the fund on a daily basis; very few VCTs are "self-run" by their directors. The investment managers goal is to invest in firms that maximises returns to its shareholders whilst abiding by the rules and regulations guiding the VCT scheme. To that end, VCTs monitor, work with, and provide expert advice and services to their investees to help increase their value - which in turn maximises returns for VCT investors.

We will provide more details on the VCT scheme and regulations - including how these regulations have evolved - in further sections and in the appendix. But for now, the main highlights of the VCT scheme and its regulations are that VCTs: must be listed on a U.K. recognised Stock Exchange, are exempt from corporation tax on any capital gains from the disposal of an investment, can only invest in firms with a permanent establishment in the U.K., carrying on a "qualifying trade" with fewer than 250 full-time equivalent employees at the time shares are issued, and gross assets of no more than £15 million before investment and £16m immediately after investment.[4] Potential investees can receive up to £5 million in VCT financing in any 12 month period with a lifetime cap of £12 million - where these sums are also inclusive of any investment via the other two government sponsored venture capital schemes mentioned earlier: EIS and SEIS.

A VCT will typically hold an investment for a period of three to seven years before looking to sell its stake in the investee. A very high percentage of the exit proceeds - subject to the VCT's investment policy and prevailing VCT scheme regulations - are re-invested into new investees. Also, VCT regulations require tax-free dividends be paid to investors where a gain is made. In very rare instances, some VCTs are set-up with a limited lifespan. These VCTs aim to exit from all of their investees, dissolve the VCT and return all capital to their investors after a defined

[4] With very few exceptions, most trades/industries are qualifying. HMRC places restrictions on industries that Her Majesty's Treasury does not consider as in need of extra financial support e.g. agriculture, real estate, financial services, oil & gas

period e.g. seven years. These limited-life VCTs typically focus on firms with guaranteed or contractual income, thus allowing for an easy exit within a defined period. We however note that with the introduction of new risk-to-capital guidelines for the VCT scheme in 2018, limited-life VCTs are now almost if not completely non-existent.

To encourage investment in VCTs, the U.K. government offers significant tax advantages to VCT investors. An investor in VCT shares - purchased at launch, or during subsequent share class issues - receives up to 30% tax relief on their VCT share subscriptions of up to a maximum of £200,000, conditional on holding the investments for a minimum of five years. In addition to the tax-free dividends mentioned earlier, capital gains from VCT investments are also free of capital gains tax. If an investor purchases VCT shares on the secondary market i.e. after they are listed on the London Stock Exchange, there is no tax-relief on the purchase, but gains from such secondary market purchases are free of capital gains tax, in addition to any dividends from the investment being tax-free. Investors exit from VCTs by selling their shares on the London Stock Exchange or participating in any share buy-back scheme offered by VCTs or both.

Clearly, and in addition to tax-free savings from Individual Savings Accounts (ISAs) and pension allowances, VCTs are an alternative for tax-efficient investing. We illustrate this point with a simple example. Assume a company with a share price of 200p pays a 10p dividend. The dividend yield is 5%. If an investor holds the shares of said company outside an ISA or pension, the net of tax yield is 3.38% for a higher-rate taxpayer and 3.1% for an additional-rate rate taxpayer, [5] assuming the £2,000 dividend allowance has been used. Analogously, if a VCT with an initial share price of 200p pays a 10p dividend, the yield is higher than 3.38% because the VCT investor gets up to 30% income tax relief, hence the net purchase cost of the share is actually 140p. A 10p dividend from VCT shares purchased at 140p results in a tax-free yield of 7.14%. To achieve an equivalent after-tax dividend yield of 7.14% on a taxable investment , a higher-rate tax-payer would need to earn a pre-tax yield of 10.6% whilst an additional rate tax-payer would need 11.5%.

[5] The tax rates are 32.5% and 38.1% respectively.

3 VCT Data

3.1 Hand-Collection of Investee Data

In this section, we detail the hand-collection and measurement of our data on U.K. firms that received VCT funding (investees). Our ultimate aim is to collate data on the annual total-assets of each investee, which we need for estimating the effect of the VCT scheme on the total-assets formation of investees: Average Treatment Effect on the Treated (ATT). There are two parts to estimating this estimand. The collected/observed total-assets values for investees and the counterfactual ("missing") total-assets values, which we will estimate with our Matrix Completion algorithm.

Our first task was to collate the names of all investees and the date they first received VCT funding, from the inception of the VCT scheme in 1995 to 2018. Data platforms have very sparse coverage of VCT data. The VCT regulator (HMRC) does not publish this information either. We scoured the Companies House Service,[6] the London Stock Exchange (LSE) website, and the Association of Investment Companies (AIC) website - to build a list of the names of all current (62) and former VCTs. Armed with this list, we sourced and gathered every semi-annual and annual report published by every VCT from the inception of the scheme till present day (2018). From these reports, we extracted details on investees: their names, registration number, and the date they received VCT funding for the first time. We focused on the first time an investee received VCT funding because we will adopt a staggered adoption of treatment set up in our matrix completion approach, which implies that once a firm is "treated", it remains in the "treated" group forever. This meant that we did not need to track the subsequent funding rounds of each investee. Once an investee receives VCT funding (treatment), it cannot "un-receive" it, it remains in the treated group forever. Our final sample contains 1,931 unique U.K. firms.[7] The staggered adoption of treatment set up has been extensively employed in the literature on causal potential outcomes. For more on staggered adoption, see Athey and Stern (2002), Athey and Imbens

[6]A digital search service that provides free access to all public information stored on the U.K. register of companies

[7]The number of firms that received VCT funding for the first time is closer to 2,000. However, due to data hand-collection difficulties - especially with regard to the exact date the investee received the VCT funding - we excluded some firms from this analysis.

(2018), and Athey et al. (2018).

We utilise the list of 1,931 investees and their registration numbers to obtain their total-assets data on the Financial Analysis Made Easy (FAME) database. FAME contains detailed financial, legal, and ownership information for public and privately incorporated firms in the U.K. and Republic of Ireland. Additionally, we collate total-assets data on 60,000 randomly selected but representative sample of the universe of U.K. firms (non-investees). This sample, in addition to our sample of 1,931 investees, will be employed in our Matrix Completion algorithm. It is also worth mentioning that our sample is free of survivorship bias - as FAME reports historical information for up to 10 years regardless of whether a firm reports financial data or not.

3.2 Measurement of Investee Data

Our data sample spans the time period 2003 - 2018 at an annual frequency. FAME data coverage starts from 2001, but we constricted our sample to start at 2003 because the 2001-2002 total-assets entries for a significant proportion of investees are missing. The data on U.K. non-investees is a FAME random sample which is representative of the universe of U.K. firms. These non-investees firm-year observations have no missing or zero values for total-assets between the periods 2003-2018. Our final sample consists of 1,931 investees plus 60,000 non-investees spanning the period 2003-2018, and contains information on each firm's: annual total-assets, date of incorporation, primary SIC code, company status, and SME indicator.

3.3 Summary Statistics: Hand-Collected Investee Data

Here, we present and analyse summary statistics on our hand-collected data on investees. First thing to note in Fig.1, is that the median size - as measured by the total-assets - of potential investees has varied over time. It ranges from approximately £8.3m in 2008 to approximately £1.7m in 2017. Also, we note that post-2015, the median size was at its lowest in all of the sample periods (between £1.7m - £2.7m). This is as a result of the 2015 rules prohibiting VCTs from investing in potential investees older than 7 years and the mandate that the potential investee must be an entrepreneurial firm with a genuine risk of loss of capital and the objective to grow and develop. This policy change helps explain why the median pre-VCT-funding size of investees has shrunk since 2015.

We next present the number of firms that received funding per annum in Fig.2. We observe that VCTs invested in a record-breaking number of firms in both 2014 and 2018. What does this mean? Did VCTs fundraise a record-breaking amount in both 2014 and 2018, and by implication, invest a record-breaking amount in both years - adopting a strategy of investing this record-breaking sum across a record-breaking number of firms i.e. increase the extensive margin. Did VCTs fundraise an average amount in 2014 and 2018, and by implication, invest an average amount, but spread this across a record-breaking large number of firms, hence the record number of new investees? We can answer this by jointly analysing our Fig.2 with column 2 of Table 2. It is clear that the extensive and intensive margin both increased. We observe that more money was raised in the periods 2013-2014 and 2014-2015 relative to the last 7-8 years. Also, 2018 was record-setting in terms of the amount of funds raised by VCTs - second behind the 2005 period. This leads us to conclude that not only did VCTs raise record-breaking amounts in both 2014 and 2018, they also invested in a record-breaking number of new firms.

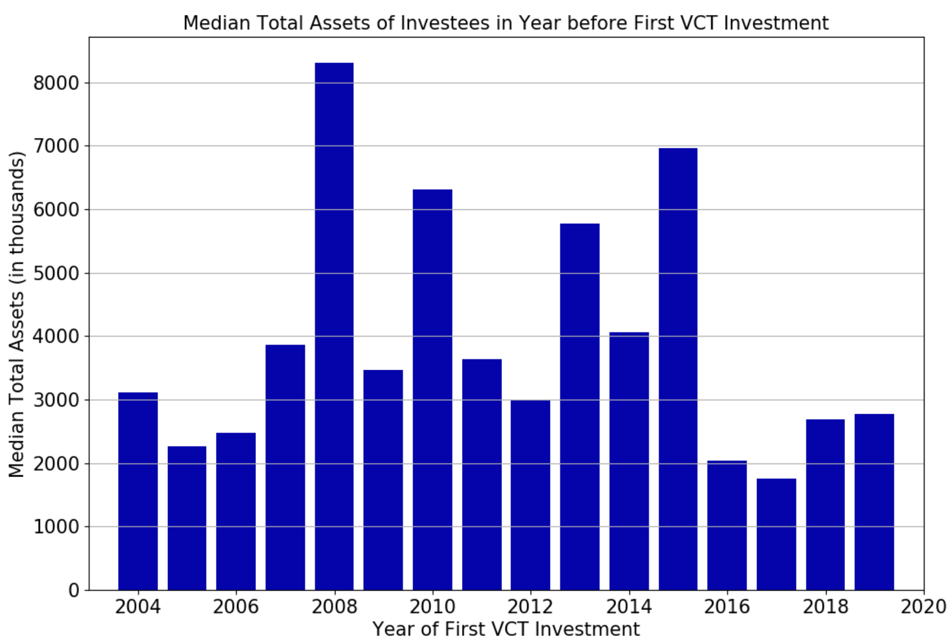

Figure 1: Median Total Assets of Investees in Year before VCT Investment

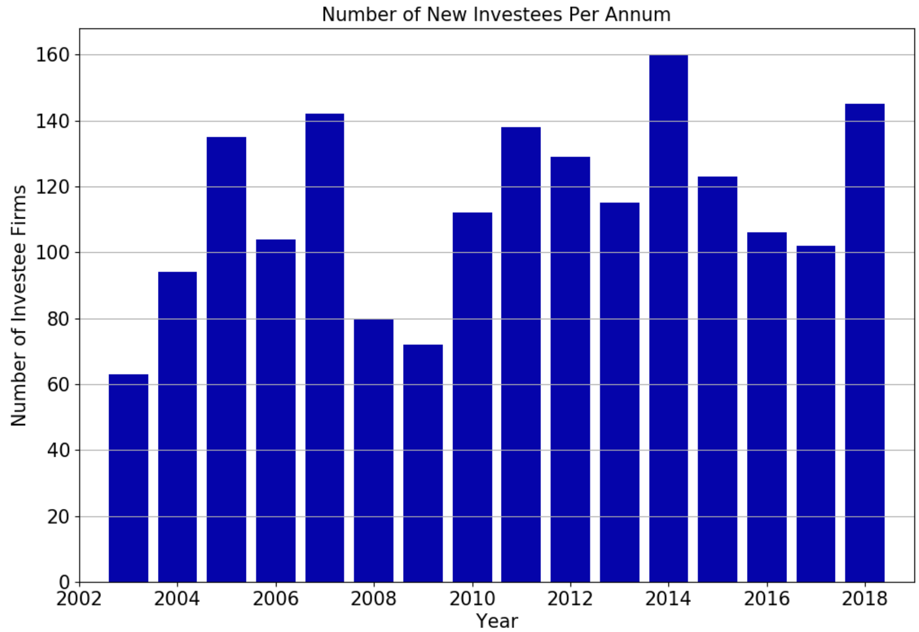

Figure 2: Number of New Investees Per Annum

Fig.3 categorises investees according to their current Companies House status. i.e. whether they are still Active or Dissolved/In Liquidation. For example, the first set of bars (blue then red) depicts the number of firms that received first-time VCT funding in 2003, categorised according to their current status (Active or Dissolved/In Liquidation). The first thing that stands out is that the majority of investees in every single cohort are still Active. In aggregate, of the 1,931 unique firms in our sample that received VCT funding for the first time between 2003-2018, 68% of them are still Active, with the remainder 32% classed as Dissolved/In Liquidation. To put these numbers in context, the Office for National Statistics (ONS) Business Demography data on the latest five year survival rate for British firms is 42.5%. Investees have seemingly out-performed the national average survival rate of new firms. However, we acknowledge that the average size of investees - as measured by the range of their recent total-assets of £1.7 million - £2.7 million, is perhaps bigger than, for instance, the average startup in the Restaurants and Mobile Food Service Activities sector, and as such, using the national average survival rate to provide context might be misleading. We thus provide a more granular context by pointing out that the national average survival rate for the Computer Programming, Consultancy and Related Activities sector is 51.4%, a sector that is synonymous with large enterprises. This survival rate is still lower than that of investees at 68%. Understanding why investees have a relatively high survival rate is an important question, and will be the subject of future research.[8]

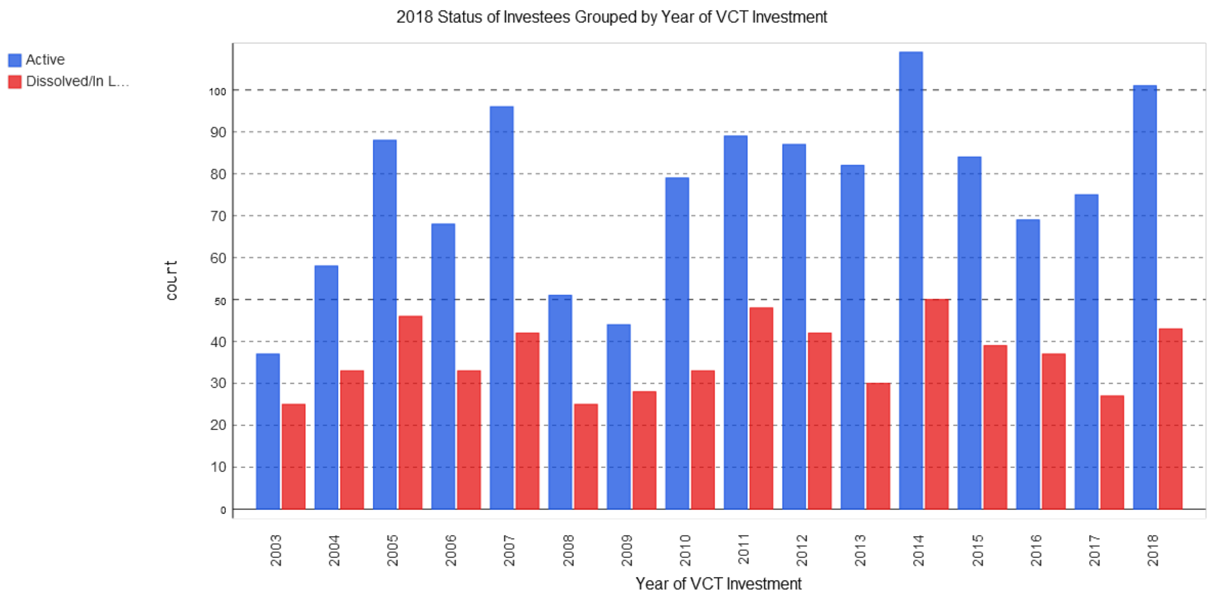

Figure 3: 2018 Status of Investees - Grouped by Year of VCT Investment

[8]The ONS Business Demography (2020) excel data file can be found at the following link: https://www.ons.gov.uk/businessindustryandtrade/business/activitysizeandlocation/datasets/businessdemographyreferencetable. The survival rate for Computer Programming, Consultancy and Related Activities is from Table 5.2a in the excel file.

3.4 Summary Statistics: HMRC Data on VCTs

Here, we present some summary statistics on VCTs raising and managing funds, collated from a recent HMRC publication (HMRC Venture Capital Trusts Statistics, 2018). In a later section, these HMRC VCT statistics will be employed in conjunction with our hand-collected data on investees, to understand how VCT policy changes and VCT fundraising drives the total-assets formation (investment) of investees.[9] In Fig.4, the first thing to note is that since the 2008-2009 period, the annual amount of funds raised by VCTs has been predominantly trending upwards. Between 2008-2018, there has been an almost 400% increase in the amount of funds raised - with this increase almost evenly spread across the period. In the Appendix: Major VCT Policy Changes, we detail major VCT policy changes over time and how they impacted VCT fundraising activity and of course their onward funding of SMEs. For now, the highlights are: the increased income tax relief from 20% to 40% in the 2004-2005 tax year explains the record setting amount of funds raised between 2004-2006; the 2017 Patient Capital Review and reduction in lifetime pension allowances was the major determinant of the sustained upward trend in fundraising since 2015-2016.

In Fig.5, the first thing to note is that the number of VCTs raising funds has almost always been less than the number of firms managing funds. VCTs do not raise funds annually. From Fig.5., we also note a consistently decreasing number of VCTs managing funds since the 2010-2011 tax period. This period coincided with the tightening of VCT rules i.e. VCT policy changes that limited the types and size of firms a VCT could invest in. Consequentially, VCTs started to merge in response to these changes and of course to achieve economies of scale. Additionally and as a further consequence of VCT policy changes and economies of scale, we note that the number of VCTs raising funds has been steadily declining since the 2013-2014 tax period, even though the amount of funds raised (Fig.4) within the same period has been on the rise. The last thing to note is the sharp fall in the number of VCTs raising funds between 2005-2006 and 2006-2007. This was due to the decrease in the income tax relief from 40% to 30% - for VCT investors.

[9] See Section 6.1

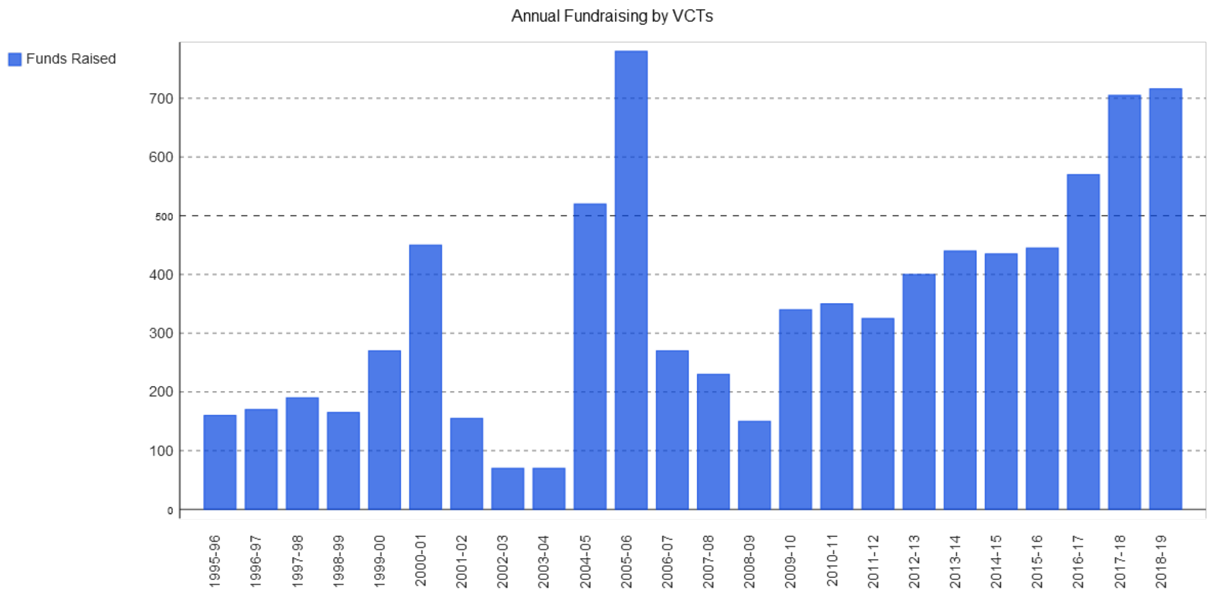

Figure 4: Annual Fundraising by VCTs (£ Million)

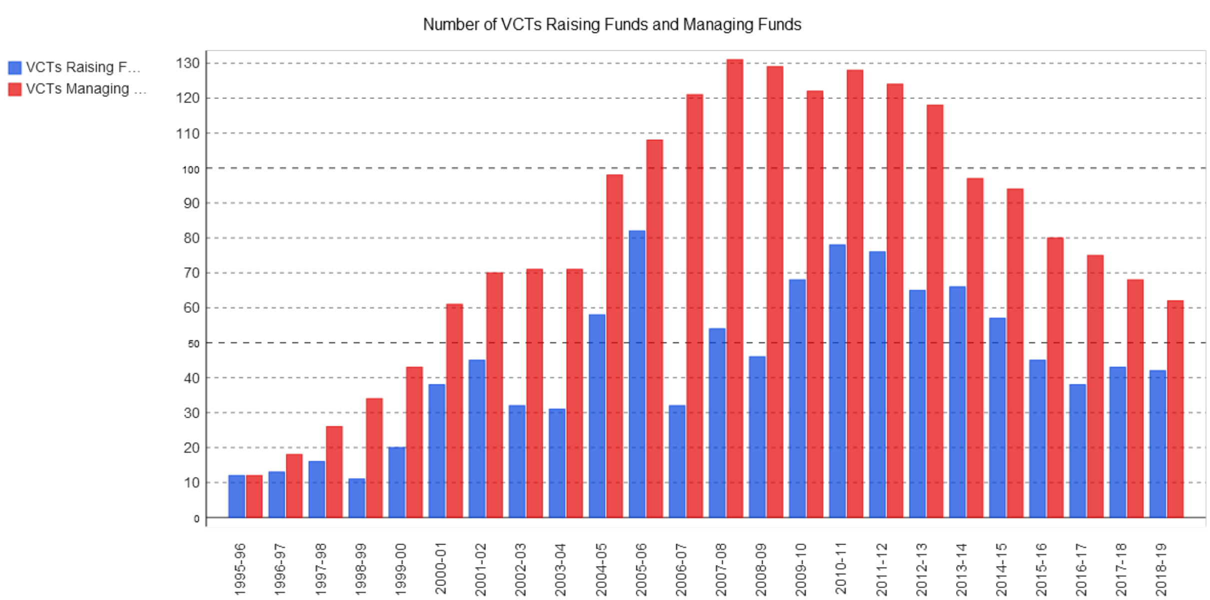

Figure 5: Number of VCTs Raising and Managing Funds

4 Average Treatment Effect on the Treated (ATT)

4.1 Set Up

In this section, we elaborate on the causal problem, illustrate how we set up our matrix of total-assets to impute counterfactual total-assets, and also set up our estimand: the average treatment effect on the treated (ATT). Recall, the Matrix Completion algorithm allows us to impute the counterfactual total-assets for investees (which in our set up is presumed "missing"), which consequently allows us to estimate the average causal effect of the VCT scheme on the total-assets formation of investees (ATT).

The set-up for our causal problem is adapted from Athey et al. (2018).

Consider an $N \times T$ matrix $\mathbf{Y}$ which denotes our panel data of total-assets for N investees observed over T periods, with typical observation Y_{it} $\forall\ i \in \{1, \cdots, N\}$ and $t \in \{1, \cdots, T\}$. This adapted setup is motivated by a causal potential outcome setting (see Athey et al. (2018); Imbens and Rubin (2015); Rubin (1974)), where at each time period, a firm is either an investee or not. We characterise this as $W_{it} \in \{0,1\}$. In other words, W_{it} is an indicator for whether a firm has received VCT funding or not. Note that in our setting, once a firm receives VCT funding, it remains in the investee group throughout the sample. Additionally, τ_{it} is an indicator for the observed and counterfactual total-assets value for an investee at period t, and is given by: $\tau_{it} \in \{0,1\}$. We now turn to laying out our estimand: the average causal effect (ATT) of the VCT scheme on the total-assets formation of investees (firms who received VCT funding). This effect is formulated as:

$$ATT = \mathbb{E}[Y_{\tau=1}|w=1] - \mathbb{E}[Y_{\tau=0}|w=1]. \tag{1}$$

To estimate this quantity for all investees, we need to impute the counterfactual ("missing") total-assets value for all investees. Given the form of our estimand (ATT), all the total-assets entries for $Y_{\tau=1}|w=1$ are observed. We want to impute the counterfactual ("missing") total-assets for $Y_{\tau=0}|w=1$. For ease of notation and uniformity with the matrix completion literature, we will interchangeably refer to our task as imputing the missing values of a partially observed matrix of total-assets $\mathbf{Y}$ or imputing the counterfactual total-assets of investees; the total-assets of investees had they not received VCT funding. With this task complete, we can estimate our average causal effect of the VCT scheme on the total-assets formation of investees: ATT.

Regarding the pattern of missing data, we know that investees received VCT funding in a staggered fashion. In other words, there is a staggered or time-varying adoption of treatment (Athey et al. (2018); Shaikh and Toulis (2019)). Essentially, this means investees received VCT funding at different periods, and in some cases, multiple times over several years. Nonetheless, once a firm receives VCT funding i.e. becomes an investee, we assume it remains an investee forever. This means that from the year it received VCT funding onward, we estimate its counterfactual total-assets (presume its total-assets is "missing"). We illustrate below:

$$
Y_{N\times T} = \begin{array}{cc}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & T \end{array} & \\
\begin{pmatrix}
\checkmark & \checkmark & \checkmark & \checkmark & \cdots & \checkmark \\
\checkmark & \checkmark & \checkmark & \checkmark & \cdots & \mathsf{X} \\
\checkmark & \checkmark & \mathsf{X} & \mathsf{X} & \cdots & \mathsf{X} \\
\checkmark & \checkmark & \mathsf{X} & \mathsf{X} & \cdots & \mathsf{X} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\checkmark & \mathsf{X} & \mathsf{X} & \mathsf{X} & \cdots & \mathsf{X}
\end{pmatrix} &
\begin{array}{l}
\text{never received VCT funding} \\
\text{received VCT funding in period T} \\
\\
\text{received VCT funding in period 3} \\
\\
\text{received VCT funding in period 2}
\end{array}
\end{array}
$$

Here, the check-mark ($\checkmark$) represents observed/pre-VCT funding total-assets values whilst the X represents "missing"/post-VCT funding total-assets values. In other words, the X represents counterfactual total-assets values which we will estimate with the matrix Completion algorithm. For instance, firm N (in the last row entry) received VCT funding in period 2 - hence the X in period 2. It may or may not have received further rounds of VCT funding in subsequent periods - up to period T. Regardless, the firm remains an investee from the moment it receives VCT funding - hence the X in periods $3, 4, \cdots, T$ as well.

5 Matrix Completion

5.1 Matrix Factorisation: Singular Value Decomposition

In this section, we develop the matrix factorisation approach which underpins our Matrix Completion algorithm, which we employ to estimate the average causal effect of the VCT scheme on the total-assets formation of investees in the U.K. The matrix factorisation approach is based on a fundamental topic in unsupervised machine learning: the recovery of a low-rank matrix from high-dimensional data or data dimensionality reduction, which helps to uncover otherwise hidden information in data. This framework is widely used in far ranging fields - from economics (Athey et al. (2018)) to computer vision (Candes and Plan (2009)). It is used to solve many popular machine learning tasks such as matrix completion (Candes and Tao (2010); Athey et al. (2018)) and robust principal component analysis (Candes et al. (2009)). This framework has also been employed in a causal panel data settings in Economics (Athey et al. (2018)), and in the building of recommender systems (Koren, Bell and Volinsky (2009)). The idea behind matrix factorisation is that the data is given in the form of a matrix Y, and we assume that the true dimensionality of the matrix (for example, the rank of the matrix) is much lower than the actual dimension of the matrix Y. This assumption can be formulated as:

$$\mathbf{Y} = WZ^{\top}, \tag{2}$$

for matrices $Y \in \mathbb{R}^{N\times T}$, $W \in \mathbb{R}^{N\times k}$ and $Z \in \mathbb{R}^{T\times k}$. If k is smaller than N and T, the rank of $\mathbf{Y}$ is k instead of N or T. Practically, this means we only store k(T + N) values of Y instead of NT values. The former being much smaller if k is chosen to be small. To illustrate, consider our panel data of investee-years given by $Y \in \mathbb{R}^{1931\times 16}$, where every row is the vector representation of one investee, every column represents the 16 years between 2003-2018, and assume that all 1,931 investees can (approximately) be considered linear combinations of only 10 different firms, i.e. k = 10. This means we can store the data on all firm-years with only 10 × (16 + 1,931) = 19,470 entries, as opposed to the NT=30,896 entries of the original dataset. This is approximately 63% of our original investee-years dataset.

There are numerous approaches to factorising matrices. In this paper, we focus on the singular value decomposition (SVD) approach; SVD generalises the concept of eigendecompositions of square matrices. It can be shown that every real matrix $Y \in \mathbb{R}^{N\times T}$ can be factorised into three matrices $U \in \mathbb{R}^{N\times N}$, $\Sigma \in \mathbb{R}^{N\times T}$ and $V \in \mathbb{R}^{T\times T}$ via

$$Y = U\Sigma V^{\top}, \tag{3}$$

where, both U and V are orthogonal matrices, i.e. $U^{\top}U = I_{N\times N}$, $UU^{\top} = I_{N\times N}$, $V^{\top}V = I_{T\times T}$ and $VV^{\top} = I_{T\times T}$, with their columns called left- and right- singular vectors of Y. In our case, where our matrix of total-assets has $N > T$, the matrix Σ is a diagonal matrix of the form:

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

The entries $\{\sigma_j\}_{j=1}^{\min(N,T)}$ are the singular values of the total-assets matrix Y, and are all non-negative i.e. $\sigma_j \geq 0 \; \forall \; j \in \{1, \cdots, \min(N,T)\}$.
Clearly, 2 is a special case of 3 i.e. $W = U\Sigma$ and $Z = V$. The SVD of our total-assets matrix Y allows us to easily compute the Frobenius norm of said matrix, given that the Frobenius norm is equivalent to the euclidean norm of the vector of singular values. Now, we can easily define our lower dimensional approximation of total-assets matrix Y, with help from its SVD.
Suppose we define a new matrix $U_k \in \mathbb{R}^{N \times T}$ as the first k columns of U. We thus have:

$$U_k U_k^\top Y = U_k U_k^\top \, U \Sigma V^\top = U_k \left(I_{k\times k} 0_{k\times(N-k)} \right) \Sigma V^\top = U \Sigma_k V^\top, \tag{4}$$

where $\Sigma_k \in \mathbb{R}^{N \times T}$ is defined as:

$$\Sigma_k = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Therefore, $Y_k = U\Sigma_k V^\top = U_k U_k^\top Y$ is a rank k approximation of our total-assets matrix Y. In actuality, it is more than a rank-k-approximation, it is the *best rank-k-approximation* in the sense of the Frobenius norm.

Theorem 5.1. (Best rank-k-approximation). This theorem is based on the Eckart-Young-Minsky theorem.

For any matrix $\hat{Y} \in \mathbb{R}^{N \times T}$ with rank $(\hat{Y}) = k$, we have:

$$\left\|Y - \hat{Y}\right\|_{FRO}^2 \geq \|Y - Y_k\|_{FRO}^2 = \left\|Y - U_k U_k^\top Y\right\|_{FRO}^2 = \sum_{j \geq k+1}^{\min(N,T)} \sigma_j^2.$$

Therefore, Y_k is the best rank-k-approximation in the sense of the Frobenius norm.

See Eckart and Young (1936) for a proof of this theorem.

5.2 The Matrix Completion Estimator

We continue with our set up adapted from Athey et al. (2018).

Given our $N \times T$ panel data/matrix of total-assets Y of investees, which we model with the form:

$$\mathbf{Y} = \mathbf{L}\ , \tag{5}$$

our goal is to find a low-rank approximation to said matrix. The first a-priori assumption that we want to make is that the investees in our matrix can be classified into types, and that the different types are less than N. Therefore, we assume that every investee in our matrix Y can be modelled as a linear combination of all investee types. Mathematically, this means that we assume that the matrix with all entries has a low-rank.

The task of finding a low-rank matrix approximation $\hat{L} \in \mathbb{R}^{N \times T}$ [10] of our total-assets matrix $Y \in \mathbb{R}^{N \times T}$ can be formulated as the convex optimisation problem:

$$\hat{L} = \underset{L \in \mathbb{R}^{N \times T}}{\arg\min} \left\{ \frac{1}{2} \|L - Y\|_{\text{Fro}}^2 \ + \ \alpha \|L\|_* \quad \text{subject to } P_\Omega L = P_\Omega Y \right\}, \tag{6}$$

where $\|\cdot\|_*$ denotes the *nuclear-norm*, which is the one-norm or the sum of the vector of singular values of Y. i.e.

$$\|L\|_* = \sum_{j=1}^{\min(N,T)} \sigma_j,$$

[10]The low-rank matrix L has rank-r where $r << \min(N,T)$ so that it is low-rank

where $\alpha > 0$ is a regularisation parameter and $\left\{\sigma_j\right\}_{j=1}^{\min(N,T)}$ denotes the singular values of L. Effectively, the *nuclear-norm* implicitly penalises the rank of the matrix $\hat{L}$ that we wish to recover. In order to ensure that the entries for which $\hat{L}$ is known matches the observed entries, we impose the constraint $P_\Omega L = P_\Omega Y$.

$P_\Omega : \mathbb{R}^{N\times T} \to \mathbb{R}^r$ denotes the projection onto the r observed entries of our total-assets matrix Y, provided by the set Ω. $P_\Omega Y$ are the known values of our total-assets matrix at these indices. To illustrate, we characterise our orthogonal projection operator P_Ω as

$$P_\Omega(Y)_{it} = \begin{cases} Y_{it}, & \text{if } (i,t) \in \Omega \\ 0, & \text{otherwise,} \end{cases}$$

and assume our incomplete total-assets matrix Y is given as

$$\mathbf{Y} = \begin{pmatrix} 1 & 4 & ? \\ ? & 2 & 7 \end{pmatrix}.$$

We know the indices $\Omega = \{(1,1),(1,2),(2,2),(2,3)\}$, and can therefore project them, i.e.

$$P_\Omega Y = (1 \;\; 4 \;\; 2 \;\; 7)^\top$$

Note that this operator is linear and its transpose operation $P_\Omega^\top : \mathbb{R}^r \to \mathbb{R}^{N\times T}$ is

$$P_\Omega^\top = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 & 0 \\ 0 & z_3 & z_4 \end{pmatrix},$$

for $z := P_\Omega Y$.

In the appendix, we derive a computationally efficient algorithm for the numerical solution of our optimisation problem 6. It is also pertinent to emphasise that 6 is a proximal mapping, a mapping we show has a simple closed form solution - see appendix.

6 The Causal Effect of the VCT Scheme on the Total Assets Formation (Investment) of Investees

We now turn to presenting our main findings, but first we re-present our estimand, the Average Treatment effect on the Treated (ATT), for which one of its inputs (the counterfactual total-assets of investees), requires the matrix completion method of imputing missing values in an incomplete matrix. The ATT is the average causal effect of the VCT scheme on the total-assets formation (investment) of investees in the U.K. Total-assets formation or investment is the change in total-assets.

Given our approximated complete matrix (Y) of annual total-assets for all investee-years between 2003-2018, wherein we reiterate that the observed entries for investees prior to receiving VCT funding are unchanged in the approximated Y matrix, the ATT is calculated as:

$$ATT = \mathbb{E}[Y_{\tau=1}|w=1] - \mathbb{E}[Y_{\tau=0}|w=1]. \tag{7}$$

where for each investees, $Y_{\tau=1}|w=1$ is its observed investment (total-assets formation or change in total-assets). $Y_{\tau=0}|w=1$ is its counterfactual investment (total-assets formation or change in total-assets). W is an indicator for whether the firm is an investee or non-investee, and τ is an indicator for the observed or counterfactual investment.

Fig.6 is a plot of our main result - also tabulated in Table 1. It depicts the annual Average Treatment effect on the Treated (annual ATT). This captures the annual average difference between the observed vs. counterfactual investment for investees. As with Fig.6, we see in Table 1, that between 2004-2007, the VCT scheme caused a substantial aggregate increase in the investment of investees (increase in the total-assets formation), from 26.40% to 49.30%. 2007 heralds the beginning of a precipitous drop in the VCT-induced investment of investees. A drop that reaches its nadir in 2009 at 30.00%. Thereafter, we see a slightly sustained rise in the causal effect of the VCT scheme on investment for investees, one that peaks in both 2011 and 2014 at 50.32% and 44.90% respectively. From 2014, we have another sustained downward trend which lasts until 2016. Thereafter, the trend reverses - increasing from its 2016 value of 38.16% to 50.60% in 2018. The Average Treatment effect on the Treated (ATT): Eq.7, is the average of the values in column 2 of Table 1. The ATT is 41%. This implies the VCT scheme caused a 41%

increase in the investment of investees in the U.K., between 2003-2018. An important inquiry into this 41% increase is: At what cost has this increase come? We will discuss this query in a subsequent section (Cost to Taxpayers). We now turn our attention to analysing the drivers of the aggregate annual investment (total-assets formation) of investees $Y_{\tau=1}|w=1$.

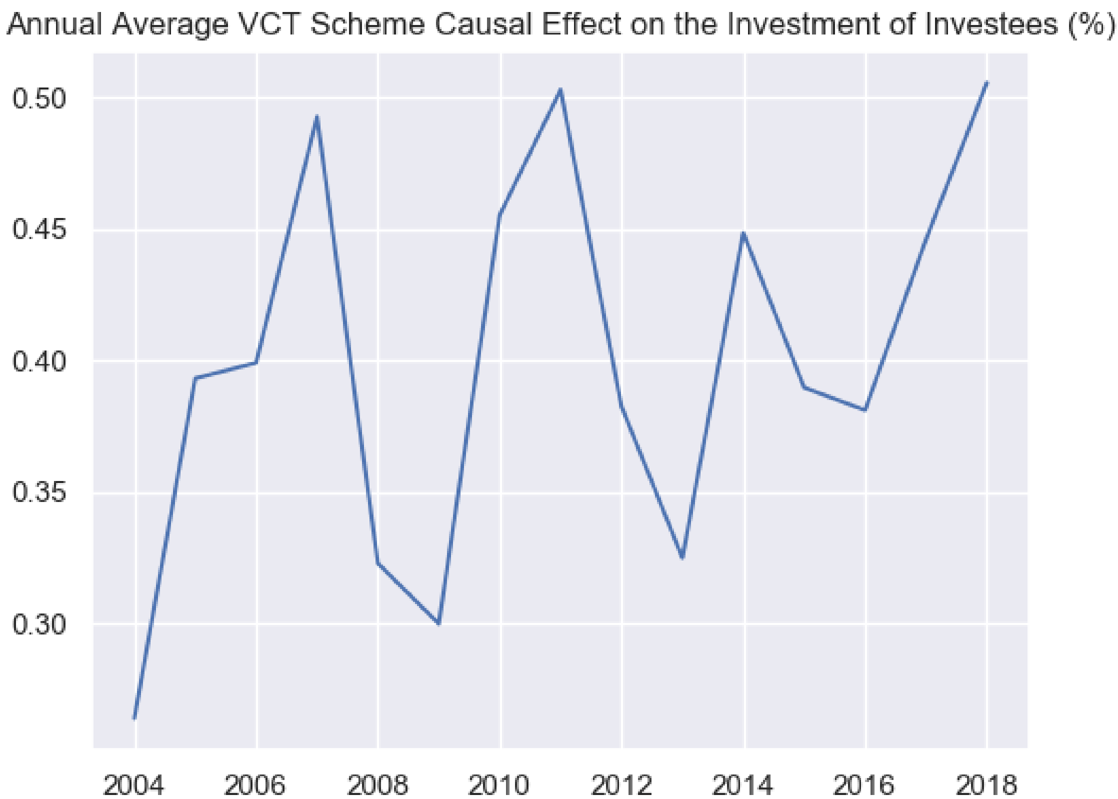

Figure 6: Annual Average VCT Scheme Causal Effect on the Investment of Investees (%)

Table 1: **Annual Average VCT Scheme Causal Effect on the Investment of Investees**

Year	ATT (%)	Number of Investees
2004	26.40	490
2005	39.33	555
2006	39.90	659
2007	49.30	726
2008	32.30	766
2009	30.00	832
2010	45.50	859
2011	50.32	900
2012	38.30	967
2013	32.50	1062
2014	44.90	1162
2015	38.98	1218
2016	38.16	1255
2017	44.61	1326
2018	50.60	1333

6.1 Aggregate Investment of Investees, VCT Fundraising and Major VCT Policy Changes

In this section, we turn to uncovering the drivers of a key component of our ATT result - the observed total-assets formation of investees $Y_{\tau=1}|w=1$. We see in Fig.7, that there is a high degree of co-movement between annual VCT fundraising (data obtained from HMRC) and the aggregate investment (total-assets formation) of investees (hand-collected data on investees) within the period 2004-2018. We now turn to linking major VCT policy changes within the period - which we document in the Appendix - to the patterns in Fig.7.

We start with Table 2, where we see a 244% aggregate increase in the amount of funds raised in 2004 and 2005 relative to the aggregate raised in the two years prior. VCT investment man-

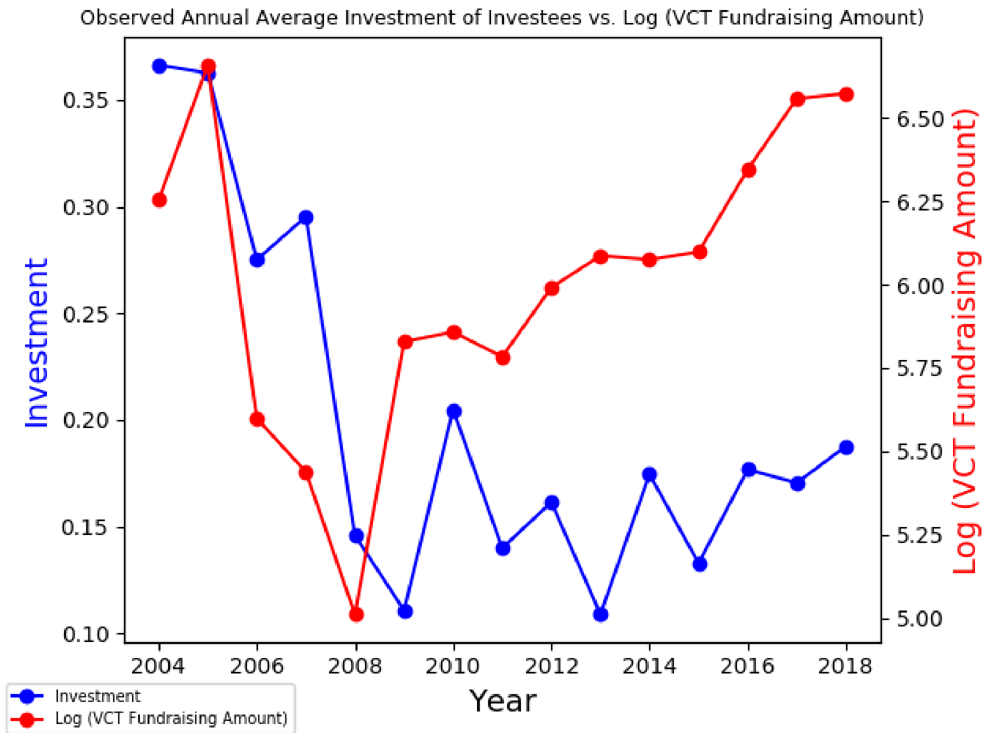

Figure 7: Observed Annual Average Investment of Investees vs. Log (VCT Fundraising Amount)

agers attributed this high level of fundraising and the concurrent high levels of investment by investees in the 2004-2006 period to the U.K. government's decision to raise the VCT income tax relief from 20% to 40% and increase the maximum amount individual investors could invest in VCTs to qualify for income tax relief - from £100,000 to £200,000 (6th April 2004). These expansionary changes to the VCT policy were only temporary, and in anticipation of their reversal, we see a significant drop in VCT fundraising activity and a depression in investment activity between 2005-2006. Between 2007-2009, we see a sustained downward trend in the aggregate investment of investees. This trend was not only driven by the financial crisis within the period, VCT investment managers report on how the 2007 VCT policy changes depressed their fundraising and investment activities thereafter. These policy changes mandated that VCT qualifying investees must be firms with fewer than 50 full-time employees and limited the amount of VCT funding a firm could raise, to a maximum of £2 million in any 12 month period. VCT investment managers also documented how this reduction in the size of qualifying investees increased the risk profile of potential investees and further depressed their investment activities as seen in Fig.7. Between 2009-2010, VCT investment managers documented their concerns about the tightened lending conditions experienced by SMEs as a result of the lingering effects of the 2008 financial crisis. They however saw this as an opportunity to fundraise and further invest in their existing portfolios, as tightened lending conditions meant VCTs were one of the few sources of working capital and expansion capital for investees. This explains the 2009-2010 upward trend in both VCT fundraising and aggregate investment of investees. We also see in Table 2, that even though fundraising in the period was at a three-year high, the number of new investees that received VCT funding was the lowest it had been since 2003 (see Fig.2). This means, VCTs raised more money relative to the last three years, but fewer new investees received said funds. Indeed, VCTs document how they viewed the tightened lending conditions as an opportunity to solidify their existing positions under favourable terms, hence a large proportion of the three-year-record-breaking newly raised funds went to existing investees. Between 2010-2013, Fig.7 depicts another downward trend in the aggregate investment activity of investees but an upward trend in VCT fundraising. VCT investment managers attribute the upward fundraising trend to the series of major VCT policy changes within the period, changes covered in the Appendix (Major VCT Policy Changes), the highlight of which centres around the reversal of the contractionary VCT policies introduced in 2007. These reversals were introduced to stimulate VCT fundraising and subsequent investment in U.K. SMEs. However, VCT investment managers were conservative in their investments. They documented their concerns about an uncertain and fragile U.K. economy. The main highlights of their concern were the sovereign debt crisis in the eurozone, upward inflationary pressures, and a sustained downward pressure on public sector spending. These reasons help explain the downward aggregate investment activity of investees trend we see in the period in Fig.7.

Between 2014-2015, we see a depression in the aggregate investment activity of investees and an imperceptible change in VCT fundraising. This was as a result of new legislation passed by the U.K. government in 2014 that prevented VCTs and their investors from refreshing income tax relief. However, from 2015 onward, we see a sustained upward trend in both VCT fundraising and aggregate investment activity of investees. These are as a result of the 2015 VCT policy changes introduced to bring the VCT scheme in line with the European Union's risk capital guidelines, as well as contemporaneous changes to U.K. government regulations surrounding other tax-advantaged investments. The highlight of the VCT policy changes were restrictions on investments that VCTs can make, particularly with respect to the age of potential investees, where potential investees were limited to firms that are less than 7 years old (ten years for knowledge intensive businesses). Investment managers documented their concerns that these policy changes will curtail their investment in Alternative Investment Market (AIM) shares; AIM shares form a significant proportion of VCT portfolio holdings. This line of reasoning is clearer when we consider that the London Stock Exchange requires that firms be at least 3 years old before they can registered on the AIM. VCT investment managers further interpreted these VCT policy changes as likely to reduce the scope of investments they could make, potentially increasing the risk profile of their portfolios. For instance, they claimed that replacing the shares of AIM firms with that of smaller unquoted firms will increase the risk profile of their portfolios.

However, there were two countervailing forces affecting VCT fundraising and aggregate investment activity of investees. On the one hand, the narrower set of investment opportunities documented by VCT investment managers could potentially depress investment activity. To paraphrase the sentiments of numerous investment managers "These new inhibitions will curtail significant drivers of growth in the U.K. SME ecosystem. They will curtail, as opposed to encourage, investment activity". On the other hand - and this sentiment was also explicitly expressed by VCT investment managers in their annual report - there is a high demand for VCTs to fundraise as a result of a reduction in the pension lifetime allowance from £1,250,000 to £1,000,000, the tapering away of pension tax allowances for high earners earning £110,000 a year or more, which can gradually reduce their annual allowance from the standard £40,000 to as low as £10,000,[11] and the launch of pension freedoms that allow for cash to be taken out of the pot for investment rather than buying an annuity. All of these factors caused VCTs to become more attractive to investors seeking additional tax-advantaged investments. The tax-advantage phenomena clearly dominated the narrower set of investment opportunities phenomena, and helps explain the upward trend we see in both VCT fundraising and aggregate investment activity of investees beginning in 2015 till the end of our sample in 2018.

Another crucial driver of the upward trend in VCT fundraising and aggregate investment activity of investees within the latter periods of our sample, especially the 2017-2018 period, was the November 2017 Patient Capital Review, in which the U.K. Government reviewed the VCT

[11] Prior to 2009, high earners could save up to £235,000 a year in a pension and receive nearly £100,000 in tax relief. As of 6th April 2016, that sum is limited to £10,000 in a pension and just £4,000 in tax relief.

scheme as part of its wider Patient Capital Review, which considered how to support innovative firms to access the finance they need to scale up. Her Majesty's Treasury published a consultation seeking views on how to increase the supply of capital to growing, innovative firms. The outcome was a number of proposed changes to the VCT regulations in an effort to refocus investment on potentially higher risk sectors that require capital (Her Majesty's Treasury Policy Paper (2017)).[12]

[12] See Appendix (Major VCT Policy Changes) for a summary of the Patient Capital Review proposals.

Table 2: **Amount of Funds Raised and Number of VCTs. Amount:Millions. Number:Actual. Data from HMRC VCT Statistics (2018)**

Year	Funds Raised	VCTs Raising Funds in the Year	VCTs Managing Funds	Income Tax Relief
	Amount	Number	Number	(%)
1995-1996	160	12	12	20%
1996-1997	170	13	18	20%
1997-1998	190	16	26	20%
1998-1999	165	11	34	20%
1999-2000	270	20	43	20%
2000-2001	450	38	61	20%
2001-2002	155	45	70	20%
2002-2003	70	32	71	20%
2003-2004	70	31	71	20%
2004-2005	520	58	98	40%
2005-2006	780	82	108	40%
2006-2007	270	32	121	30%
2007-2008	230	54	131	30%
2008-2009	150	46	129	30%
2009-2010	340	68	122	30%
2010-2011	350	78	128	30%
2011-2012	325	76	124	30%
2012-2013	400	65	118	30%
2013-2014	440	66	97	30%
2014-2015	435	57	94	30%
2015-2016	445	45	80	30%
2016-2017	570	38	75	30%
2017-2018	705	43	68	30%
2018-2019	716	42	62	30%
Total	8,375			

6.2 Additional Results

In this section we employ our hand-collected data of investees and FAME data of non-investees[13] to show how the investment pattern of investees compares to that of non-investees ("control group"). The aim is to understand the patterns behind the counterfactual ("missing") total-assets value imputed by our Matrix Completion algorithm, and used in the calculation of our ATT. In Fig.8, we plot the observed average investment for investees vs the observed average investment for our representative random sample of 60,000 non-investees in the U.K. We observe an ostensible difference between the investment patterns of investee vs non-investees. Not only do investees - in the aggregate - invest at a much higher rate than non-investees, we also observe divergent aggregate patterns since 2009. For instance, from 2013 onward, the aggregate investment trend of investees (red line) has been steadily rising, whereas that of non-investees has steadily fallen. We however note the very similar declining investment trends for both investee and non-investees between the period 2004-2009.

We note that plotting averages can mask other patterns in the data for non-investees, especially as the non-investees range in size from the smallest firms with less than £1,000 in total-assets, to the largest with £20 billion in total-assets. To allay this concern, we repeat Fig.8 with one crucial change. We plot in Fig.9, the investment of investees vs the investment of non-investees in the top decile of investment among non-investees. We see a similar pattern in Fig.9, that we see in Fig.8, albeit with different levels of investment, where the top decile non-investees are also dis-investing but their aggregate investment remains positive, whereas the dis-investment trend in Fig.8, is largely negative. Between 2004-2013, the top decile non-investees had an ostensibly similar trend in their investment pattern relative to investees, although we see that the downward trend for investees is interspersed with a few periods of upward trends (2006-2007, 2009-2010). However, from 2013, we see that the aggregate investment pattern of these non-investees continues to decline - a decline that carries on to the end of the sample in 2018. On the other hand, we see an upward trend in the aggregate investment rate for investees beginning in 2013 till the end of the sample in 2018. We have already tied this increased investment rate to the major VCT policy changes and VCT fundraising in the period, so we will not belabour the point.

For completeness, we also repeat the same exercise for the non-investees in the bottom instead of

[13] Our hand-collected data of 1,931 investees plus the FAME downloaded data of 60,000 non-investees

the top decile of investment as depicted in Fig.10. This plot is also very interesting in the dynamic it depicts. There is an ostensibly similar trend in the aggregate investment pattern of investees and the bottom decile non-investees between 2003-2013, sometimes with a lag. However, and similar to the top decile non-investees, we see that the bottom decile non-investees have been continuously dis-investing from 2013 till the end of our sample in 2018. In summary, what we see from all three figures is that from the 2013 period till the end of our sample in 2018, the aggregate investment pattern of investees was trending upward while that of non-investees (control group) was trending downward, which further emphasise the impact of the VCT scheme, and our uncovering of a significant ATT of 41%.

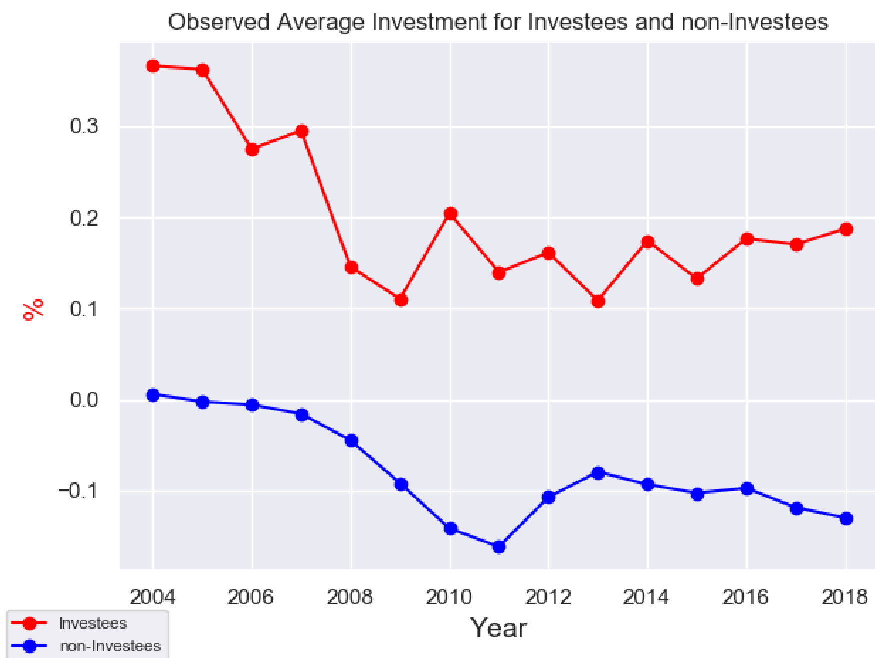

Figure 8: Observed Average Investment for Investees and non-Investees

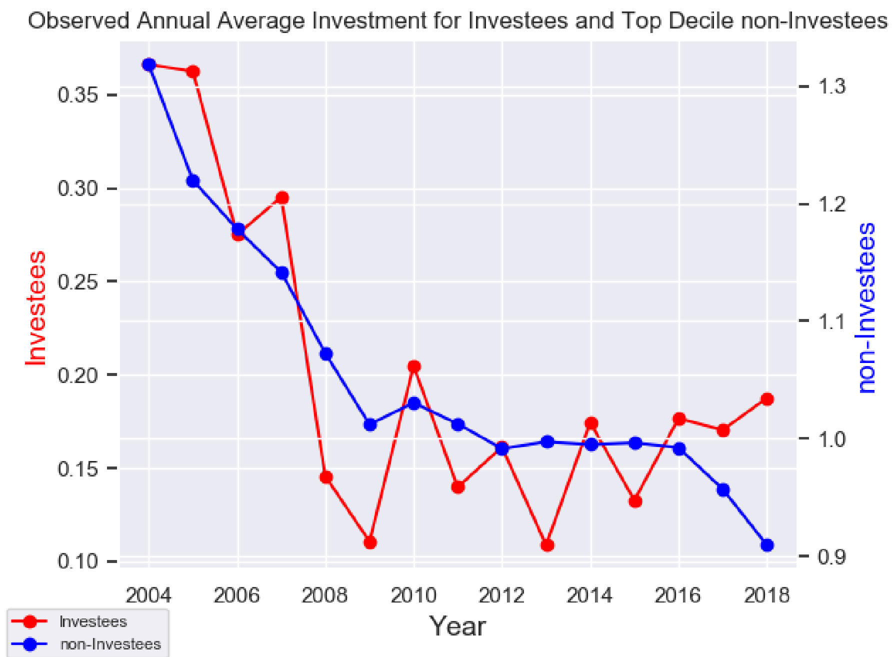

Figure 9: Observed Annual Average Investment for Investees and Top Decile non-Investees

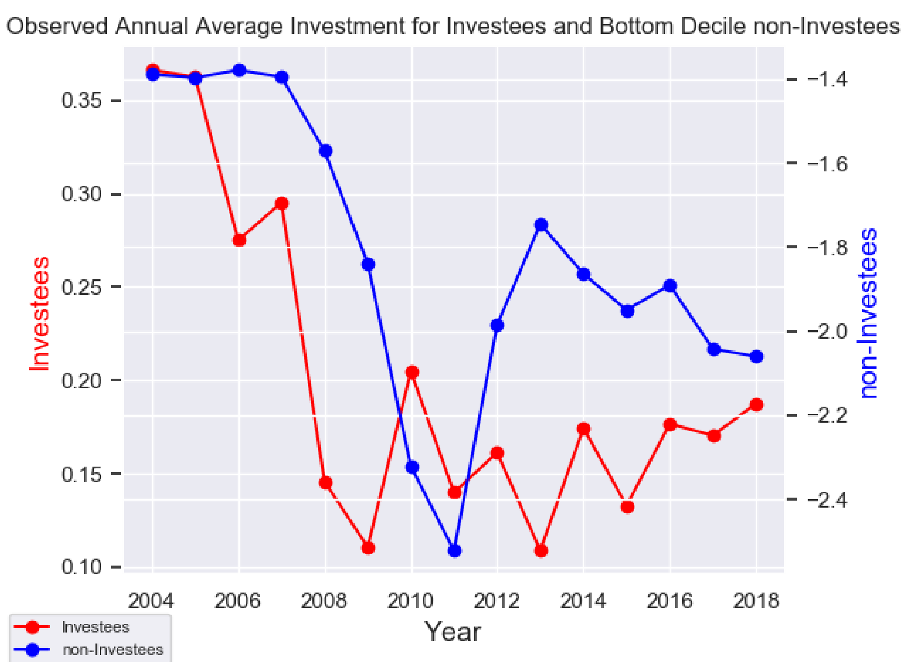

Figure 10: Observed Annual Average Investment for Investees and Bottom Decile non-Investees

6.3 Cost to Taxpayers

From the HMRC VCT data in Table 2, we observe that approximately £8.4 billion pounds has been raised by over 200 VCTs since inception of the VCT scheme in 1995.[14] These funds have funded the activities of SMEs and increased the total-assets formation (investment) of investees in the U.K. Specifically, we have shown that these funds have had a measurable positive impact on investment in the U.K. via its effect on investees - a 41% average increase in the investment of investees. However, we also know that VCT investors receive tax breaks such as: 30% upfront income tax relief, tax-free dividends, and exemption from capital gains tax. These tax breaks come at a non-trivial cost to the U.K. tax-payer, as it involves the actual reduction of a VCT subscribers tax bill as detailed in section (All About VCTs) and illustrated in the appendix. To further illustrate, in fiscal year 2017-2018, the subsidy expenditure for the VCT scheme was £201 million.[15] This figure is extremely conservative as it does not take into account investors making income tax relief claims through Self-Assessment nor does it consider investors making claims through other systems e.g. PAYE. Also, it does not include other tax reliefs and exemptions available through the VCT scheme such as capital gains and dividend-tax exemptions. For comparisons sake, this conservative £201 million in subsidy expenditure is over half of the total managed expenditure, [16] over the same period, for the Department for International Trade (HM Treasury Public Expenditure Statistical Analysis (2019)), which was £394 million.

In Table 3, we present HMRC data on the amount of investment on which relief was claimed on an annual basis between 2015-2018. We note that HMRC emphasises that the investor-level information in Table 3 was prepared using Self Assessment (SA) returns. Thus, the information in Table 3 will not cover investors making income tax relief claims through other channels (e.g. PAYE) or not making any claims. However, we know these omissions are small - because we can

[14] To avoid duplication, we do not calculate the total for VCTs raising funds in column 3, Table 2, as VCTs can raise funds in multiple tax-years.

[15] £607 million - total from the last column in Table 3, which is the the amount of investment on which VCT investors claimed tax relief in the 2017-2018 fiscal year - multiplied by 30% tax relief.

[16] The total managed expenditure is the total amount the government spends. This is split up into: departmental budgets – the amount that government departments have been allocated to spend, also known as Departmental Expenditure Limits, and money spent in areas outside budgetary control – all spending that is not controlled by a government department and includes welfare, pensions and things such as debt interest payments, also known as Annually Managed Expenditure.

compare them with the amount of funds raised by VCTs in the corresponding year (Table 2).

Table 3: **Venture Capital Trusts**

Income Tax Relief; Distribution of Investors and Amount of Investment on which Relief was Claimed from 2015-16 to 2017-18.

Data from HMRC VCT Statistics (2018).

(Upper Limit: £)	2015-2016		2016-2017		2017-2018	
	Investors	Amount of Investment (£ million)	Investors	Amount of Investment (£ million)	Investors	Amount of Investment (£ million)
1,000	1,240	1	1,075	0	1,230	1
2,500	630	1	775	1	815	1
5,000	1,365	6	1,615	7	1,720	7
10,000	2,545	22	2,800	24	3,470	29
15,000	1,195	16	1,440	19	1,700	22
20,000	1,285	24	1,490	28	1,875	36
25,000	775	18	910	21	1,140	27
50,000	2,155	83	2,550	98	3,395	130
75,000	640	40	775	48	1,020	64
100,000	635	60	670	62	995	93
150,000	330	41	410	51	535	66
200,000	620	122	725	142	1,000	194

6.4 Comparison of Matrix Completion and Difference-in-Differences Estimators

Our focus in this section is twofold. We illustrate in Table 4 and Figure 11, the mechanics of our Difference-in-Differences estimator, which in turn implicitly emphasises the ability of our Matrix Completion estimator at alleviating the selection bias detailed in the introduction section. But first, we compare the imputation accuracy or performance of our Matrix Completion estimator against a Difference-in-Differences (DID) estimator. We follow the general procedure in Athey et al. (2018), whereby they compare the imputation accuracy of their Matrix Completion method against four other estimators, including the DID and synthetic control estimators. For this exercise, we use the data for non-investees with N = 60,000, T = 16. Note that in the original data set there are 61,931 firms, where 1931 are investees (VCT funded), and will be excluded from this analysis. Thereafter we artificially allocate some non-investees and time periods to be VCT funded (pseudo treated/pseudo investees), and compare counterfactual/predicted total-assets values for these pseudo investee/time-periods against their actual total-assets values. Our setting is one with staggered adoption where we randomly designate non-investees as pseudo investees, with the date of VCT funding (treatment date) varying randomly among these pseudo investees. Once you receive VCT funding, you stay in the VCT funded group. In other words, once you receive treatment, you remain a treated firm. Our task is to utilise our Matrix Completion and a DID estimator to predict/impute counterfactual total-assets values for these pseudo investee/time-periods and then compare the predicted/counterfactual total-assets values for these pseudo investee/time-periods against their actual total-assets values. We compare the root-mean-squared-error (RMSE) of both algorithms on values for the pseudo investee (time, period) pairs. As with Athey et al. (2018), our aim is not necessarily to pinpoint the right or wrong algorithm. We simply want to uncover which algorithm works best in our setting where investees received VCT funding at staggered periods. We find that our Matrix Completion estimator has a superior performance with a normalised RMSE of 0.10 whereas the DID estimator has a lower performance with a normalised RMSE of 0.14. The increased performance of our MC estimator is attributable to its use of additional observations i.e. pre-VCT-funding total-assets of the pseudo investees. This finding is in line with the findings in Athey et al. (2018), where they employ several illustrations and show that their Matrix Completion estimator is superior to 4 different estimators (including DID and synthetic control) under a variety of treatment settings.

We now turn to showing in Table 4 and Figure 11, the mechanics of our Difference-in-Differences estimator, which implicitly emphasises the superior ability of our Matrix Completion algorithm at alleviating the potential selection bias detailed in the introduction. Athey et al. (2018) detail how the Matrix Completion algorithm exploits both the patterns in the pre-VCT-funding total-assets observations of investees, and those in the entire 60,000 non-investees (control group), which implicitly alleviates the selection bias discussed earlier. The potential selection bias arises from VCTs investing in investees that are superior to non-investees along several dimensions that are unobserved in the data, which in turn causes the VCT funding of an investee to be endogenous, and thus the estimated ATT will be biased upwards relative to the VCT scheme's actual causal effect on total-assets formation. By exploiting all patterns in the data, most especially, the pattern in the pre-VCT-funding total-assets observations of investees, the Matrix Completion estimator's estimated counterfactual total-assets are not based on the pattern in any particular subset of the non-investees (control group), but on all of the patterns in all of the data - both investees and non-investees (control group). Whereas, with a simple parametric version of a Difference-in-Differences estimator, the estimator imputes the counterfactual total-assets of investees with the aid of control firms (non-investees) with identical lagged total-assets formation or investment (parallel trends). The approach centres around regressing the relevant periods total-assets on the lagged total-assets and then employing the regression estimates to predict the missing total-assets, which Athey et al. (2018) refer to as horizontal regression. In other words, with the horizontal regression, the researcher makes an ex-ante choice on what patterns in the data to exploit whereas with the Matrix Completion approach, the researcher allows the data to determine what patterns are exploited.

We now turn to illustrating how employing the Difference-in-Differences estimator which entails an ex-ante choice on what patterns in the data to exploit, potentially exacerbates the endogenous selection issue. To begin, we restrict our 60,000 non-investees (control group) to those with total-assets less than £16 million - as per VCT rules for the potential size of an investee, which reduces our non-investees (control group) dataset to 59,540 observations. We then sort and split them into 10 groups based on the following: we calculate the average growth rate of total assets (average investment) for each of our 59,540 non-investees (control group) during the sample period

(2003-2018). We thereafter sort the 59,540 non-investees (control group) in ascending order of their average investment. Finally, we split them into ten groups, which means non-investees with the lowest average investment rate over the 2003-2018 period are in Decile 1, and those with the highest average investment rate over the 2003-2018 period are in Decile 10. Each group (5,954) of non-investees then serves as the control group for our 1,931 investees. We then employ our Difference-in-Differences estimator to estimate ten ATT's based on the 7,885 (1,931 + 5,954) data for investees and each decile of non-investees. Our results are depicted in both Table 4 and Figure 11. When we choose Decile 1 as our control group (non-investees with the lowest average investment rate between the period 2003-2018), we find that the VCT scheme has the greatest causal effect on investees, relative to choosing any other Decile. Conversely, when we choose Decile 10 as our control group (non-investees with the highest average investment rate between the period 2003-2018), we find that the VCT scheme has the lowest causal effect on investees, relative to choosing any other Decile. This exercise emphasises that with the Difference-in-Differences estimator, the ATT we uncover is driven by the ex-ante choice we make on what control group to employ in the estimation. By allowing the Matrix Completion algorithm choose - in a data driven manner - what patterns in the data to exploit (what control group to use), we sidestep the problem of having to ex-ante choose a control group, which implies ex-ante choosing the ATT, which potentially exacerbates the endogenous selection bias issue prevalent in a causal study like ours. For completeness, we also employ our Difference-in-Differences estimator to estimate the ATT, without splitting into deciles, our data of 59,540 observations for non-investees (control group). We find that the ATT is 26.42%.

Table 4: **Difference-in-Differences: Average VCT Scheme Causal Effect on the Investment of Investees.**

Each ATT estimate is estimated with a Difference-in-Differences estimator and a different control group. The data of total-assets for non-investees (control group) is first restricted to non-investees (control group) with assets less than £16 million, then discretized into 10 groups, in ascending order of average growth rate over the sample period, with each group representing a control group for the investees.

Decile	ATT (%)
Decile 1	72.01
Decile 2	46.77
Decile 3	31.04
Decile 4	23.09
Decile 5	19.14
Decile 6	18.70
Decile 7	18.66
Decile 8	17.91
Decile 9	13.78
Decile 10	1.68

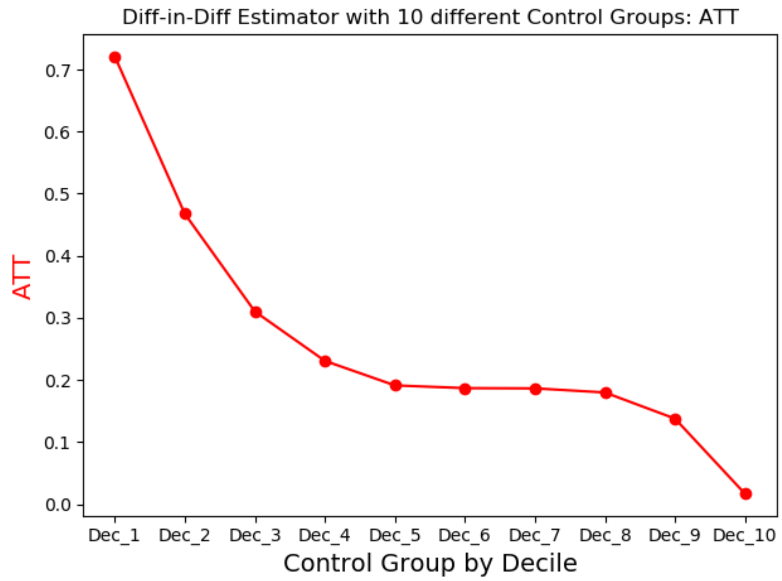

Figure 11: Difference-in-Differences Estimator with 10 different Control Groups: ATT

7 Conclusion

In this study, we sought to deepen our understanding of the VCT scheme and estimate the causal effect of the VCT scheme on the total-assets formation (investment) of investees in the U.K., between 2003-2018. We hand-collected data from all former and the current 62 VCTs operating in the U.K. Specifically, to estimate the causal effect of the VCT scheme on the total-assets formation (investment) of investees in the U.K., between 2003-2018, we hand-collected data on all investees that ever received VCT funding since inception of the VCT scheme in 1995.

We thereafter adapted and employed a Matrix Completion estimator to estimate our causal effect. This estimator is adapted from Athey et al. (2018) and has intuitive computational properties which helps alleviate the potential selection bias issue arising from estimating causal effects. We found that between 2003-2018, the causal effect of the VCT scheme on the investment of investees - the Average Treatment effect on the Treated (ATT) - was 41%. We then employed the Root Mean Square Error (RMSE) to compare the accuracy of our Matrix Completion (MC) estimator vs. a standard Difference-in-Differences (DID) estimator, at imputing missing total-assets for investees. We found that our MC estimator outperformed the DiD estimator with

RMSE's of 0.10 and 0.14 respectively.

This study contributes to two broad spheres in economics. Firstly, our results add to the literature on the importance of venture capital funding for the growth of SMEs, and is consistent with findings in Gonzalez-Uribe and Paravisini (2019), Gompers, Gornall, Kaplan and Strebulaev (2020), and Iliev and Lowry (2020), who all detail the importance of VC funding for investees. Finally, our results are practically relevant for policy makers. The insights and results we provide can serve as a template to bolster the recommendations of the Patient Capital Review - in light of the current pandemic and its adverse impact on SMEs.

A Bregman Proximal Method

A.1 Set-Up

Let $\hat{w} = \arg\min_{w \in \mathbb{R}^N} \left\{ \frac{1}{2s} \|Xw - y\|^2 \right\}$ be the compact notation for a linear regression problem. We then have the corresponding optimality condition $\nabla E(\hat{w}) = 0$ which reads as

$$X^\top X \hat{w} = X^\top y \,, \tag{8}$$

which is also known as the normal equation associated with $X\hat{w} = y$

A.2 Bregman Proximal Method

If $X^\top X$ is invertible, we can solve (8) for $\hat{w}$ with any of the numerous algorithms that are used to numerically solve linear systems of equations. However, if N is very large, the oft-used algorithms that solve (8) to exacting numerical accuracy may require substantial computational time and large memory requirements.

We can however settle on approximate solutions of (8) by employing iterative algorithms such as gradient descent - which is an iterative procedure of the form

$$w^{k+1} = w^k - \tau \nabla E(w^k), \tag{9}$$

for some energy E, step-size parameter $\tau > 0$, and an initial value $w^0 \in \mathbb{R}^N$. For example, when $E(w) = \frac{1}{2s}\|Xw - y\|^2$, gradient descent reads as

$$\begin{aligned} w^{k+1} &= w^k - \frac{\tau}{s} X^\top \left(Xw^k - y \right), \\ &= \left(I - \frac{\tau}{s} X^\top X \right) w^k + \frac{\tau}{s} X^\top y. \end{aligned} \tag{10}$$

(10) is elegant in its simplicity. Iteratively solving (10) simply requires the computation of matrix multiplications and simple arithmetic operations. Additionally, with an algorithm like (9), we can deal with minimisation problems more generic than minimising the mean squared error (MSE).

Definition A.1 (Sub-differential)**.** Let $E : \mathbf{C} \subset \mathbb{R}^n \to \mathbb{R}$ be a convex and continuous function. Its sub-differential ∂E is characterised as the set:

$$\partial E(\upsilon) := \{ g \in \mathbb{R}^n \mid E(w) - E(\upsilon) \geq \langle g, w - \upsilon \rangle, \ \forall \ w \in \mathbb{R}^n \}.$$

The elements $g \in \partial E(\upsilon)$ are known as sub-gradients.

Definition A.2 (Bregman distance)**.** Let $E : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, i.e. $\nabla E(w)$ exists $\forall\, w \in \mathbb{R}^n$ and is continuous. Then its corresponding Bregman distance $D_E : \mathbb{R}^n \times \mathbb{R}^n$ is defined as

$$D_E(u, \upsilon) := E(u) - E(\upsilon) - \langle\, \nabla E(\upsilon),\, u - \upsilon\, \rangle.$$

$\forall$ arguments $u, \upsilon \in \mathbb{R}^n$. We must emphasise that Bregman distances - defined in terms of a strictly convex function - are not necessarily distances in the sense of a metric. They are a statistical distance when the points are interpreted as probability distributions i.e. data-set of observed values. Bregman distances or divergences describe the distance of a function E at point u to its linearisation around υ, and are non-negative if and only if E is convex. A common Bregman distance is the squared Euclidean distance.

We now turn to deriving when and under what conditions (9) actually converges, what it converges to, and how quickly it converges. To maintain generality, we will consider a generalisation of gradient descent known as Bregman proximal method (BPM). This algorithm is based on Definition A.1 and an iterative procedure outlined in Algorithm 3 below.

Algorithm 1: Bregman proximal method

Specify: Energy function E: $\mathbb{R}^N \to \mathbb{R}$, Bregman function J: $\mathbb{R}^N \to \mathbb{R}$, index K

Initialise: $w^0 \in \mathbb{R}^N$

Iterate:

for $K = 0, \cdots, K-1$ **do**

  $w^{k+1} = \arg\min_{w \in \mathbb{R}^N} \left\{ E(w) + D_J(w, w^k) \right\}$;

end

return w^K.

To understand how the BPM is supposed to help us minimise an energy E such as $E(w) = \frac{1}{2s}\|Xw - y\|^2$, we must emphasise that the choice of J is critical. For instance, if we choose $J : \mathbb{R}^N \to \mathbb{R}$ with $J(w) := \frac{1}{2}\|w\|^2$, then solving the minimisation step in Algorithm 3 is just as difficult as minimising E itself. Thus, for our choice of J, we choose

$$J(w) := \frac{1}{2\tau}\|w\|^2 - E(w),$$

where $\tau > 0$ is a positive scalar. Computing the corresponding Bregman distance yields

$$\begin{aligned} D_J(w^{k+1},\, w^k) &= \frac{1}{2\tau}\left\|w^{k+1} - w^k\right\|^2 - D_E(w^{k+1},\, w^k), \\ &= \frac{1}{2\tau}\left\|w^{k+1} - w^k\right\|^2 - E(w^{k+1}) + E(w^k) + \langle \nabla E(w^k),\, w^{k+1} - w^k\rangle. \end{aligned}$$

Inserting our computed Bregman distance into the minimisation step in Algorithm 3 yields:

$$\begin{aligned} w^{k+1} &= \underset{w \in \mathbb{R}^N}{\arg\min} \left\{ E(w) + D_J(w,\, w^k) \right\}, \\ &= \underset{w \in \mathbb{R}^N}{\arg\min} \left\{ E(w^k) + \langle \nabla E(w^k),\, w - w^k\rangle + \frac{1}{2\tau}\left\|w - w^k\right\|^2 \right\}, \\ &= \underset{w \in \mathbb{R}^N}{\arg\min} \left\{ \langle \nabla E(w^k),\, w\rangle + \frac{1}{2\tau}\left\|w - w^k\right\|^2 \right\}. \end{aligned}$$

The objective function $L^k(w) := \langle \nabla E(w^k),\, w\rangle + \frac{1}{2\tau}\left\|w - w^k\right\|^2$ is convex and differentiable with gradient $\nabla L(w) = \nabla E(w^k) + \frac{1}{\tau}(w - w^k)$. Thus, the global minimiser can be obtained via $\nabla L(w^{k+1}) = 0$, which yields (9). Gradient descent is summarised in Algorithm 4 below.

Algorithm 2: Gradient Descent

Specify: Differentiable, convex function E: $\mathbb{R}^N \to \mathbb{R}$, step-size $\tau > 0$, index K

Initialise: $w^0 \in \mathbb{R}^N$

Iterate:

for $K = 0, \cdots, K-1$ **do**

  $w^{k+1} = w^k - \tau \nabla E(w^k)$;

end

return w^K.

The pertinent question now is this: Does Algorithm 3 (and by implication Algorithm 4) converge to a minimiser of the objective function E ? If it does, under what conditions does it converge ?

Theorem A.1. (Convergence of Algorithm 3). Let $E : \mathrm{C} \subset \mathbb{R}^N \to \mathbb{R}$ and $J : \mathrm{C} \subset \mathbb{R}^N \to \mathbb{R}$ be convex and continuously differentiable functions. Suppose $\hat{w}$ denotes a global minimiser of E.

Thus, the iterates of Algorithm 3 satisfy

$$E(w^K) - E(\hat{w}) \leq \frac{D_J(\hat{w}, w^0) - D_J(\hat{w}, w^K)}{K}, \tag{11}$$

and therefore $\lim_{K\to\infty} E(w^K) = E(\hat{w})$.

Before we lay out the proof of Theorem A.1, we verify the following intermediate result.

Lemma B. We adopt the same assumptions as in Theorem A.1, and suppose w^* is defined as $w^* := \arg\min_{w\in\mathbb{R}^N}\{E(w) + D_J(w, \overline{w})\}$. Consequentially, the following identity holds:

$$E(w^*) \; + \; D_E(w, w^*) \; + \; D_J(w, w^*) \; + \; D_J(w^*, \overline{w}) \; = \; E(w) \; + \; D_J(w, \overline{w}).$$

Proof. Assume we can characterise w^* via the optimality condition:

$$0 = \nabla E(w^*) + \nabla J(w^*) - \nabla J(\bar{w}).$$

Taking an inner product with $w^* - w$ then yields:

$$\begin{aligned}
0 &= -\langle \nabla E(w^*), w - w^* \rangle - \langle \nabla J(w^*) - \nabla J(\bar{w}), w - w^* \rangle, \\
&= D_E(w, w^*) - E(w) + E(w^*) - \langle \nabla J(w^*), w - w^* \rangle + \langle \nabla J(\bar{w}), w - w^* \rangle, \\
&= D_E(w, w^*) - E(w) + E(w^*) + D_J(w, w^*) - J(w) + J(w^*) + \langle \nabla J(\bar{w}), w - w^* \rangle, \\
&= D_E(w, w^*) - E(w) + E(w^*) + D_J(w, w^*) - J(w) + J(w^*) + \langle \nabla J(\bar{w}), w - \bar{w} + \bar{w} - w^* \rangle, \\
&= D_E(w, w^*) - E(w) + E(w^*) + D_J(w, w^*) - D_J(w, \bar{w}) + D_J(w^*, \bar{w}),
\end{aligned}$$

which rounds off the proof. □

Proof. **Proof of Theorem A.1** By employing Lemma B for $w^* = w^{k+1}, \bar{w} = w^k$, and $w = \hat{w}$, we have:

$$\begin{aligned}
E(\hat{w}) + D_J(\hat{w}, w^k) &= E(w^{k+1}) + D_E(\hat{w}, w^{k+1}) + D_J(\hat{w}, w^{k+1}) + D_J(w^{k+1}, w^k), \\
&\geq E(w^{k+1}) + D_J(\hat{w}, w^{k+1})
\end{aligned}$$

given the convexity of E and J, which implies $D_E(\hat{w}, w^{k+1}) \geq 0$ and $D_J(w^{k+1}, w^k) \geq 0$.
Therefore, we have $E(w^{k+1}) - E(\hat{w}) \leq D_J(\hat{w}, w^k) - D_J(\hat{w}, w^{k+1})$.
Summing from $k = 0, \cdots, K-1$ then leads to

$$\sum_{k=0}^{K-1} E(w^{k+1}) - KE(\hat{w}) \leq D_J(\hat{w}, w^0) - D_J(\hat{w}, w^K). \tag{12}$$

We can also apply Lemma B for $w^* = w^{k+1}, \bar{w} = w^k$, and $w = \hat{w}$ to obtain:

$$
\begin{aligned}
E(w^k) + \underbrace{D_J(w^k, w^k)}_{=0} &= E(w^{k+1}) + D_E(w^k, w^{k+1}) + D_J(w^k, w^{k+1}) + D_J(w^{k+1}, w^k), \\
&\geq E(w^{k+1}),
\end{aligned}
$$

given the convexity of E and J, which implies $D_E(w^k, w^{k+1}) \geq 0$, $D_J(w^k, w^{k+1}) \geq 0$, and $D_J(w^{k+1}, w^k) \geq 0$.

We can thus conclude $E(w^{k+1}) \leq E(w^k)\ \forall\ k = 0, \cdots, K-1$, especially $KE(w^k)\ \leq\ \sum_{k=0}^{K-1} E(w^{k+1})$.
If we plug this inequality into (12), it implies 11. Given J is convex, we can also estimate:

$$
E(w^K)\ \leq\ E(\hat{w}) + \frac{D_J(\hat{w}, w^0)}{K}
$$

for a positive constant $D_J(\hat{w}, w^0)$ independent of K, thus concluding both $\lim_{K\to\infty} E(w^K) = E(\hat{w})$ and the proof.

As a little aside, it is clear that showing the convexity of E and J is sufficient to prove convergence of the objective E.

□

Remark 1. It is pertinent to emphasise that Theorem A.1 does more than guarantee the convergence of Algorithm 3. It also gives us a rate of convergence. This rate is $1/K$, which in convex optimisation is emphasised with the big O-notation, i.e.

$$
E(w^k) - E(\hat{w}) = O(\frac{1}{K}),
$$

which means the left-hand-side is proportional to $1/K$. To illustrate, assume $D_J(\hat{w}, w^0) = 10$, then we will require $K = 1000$ iterations to ensure $E(w^k) - E(\hat{w}) \leq 10^{-2}$ according to Theorem A.1.

C VCT Tax Benefits: Illustrations

In this paper, we do not highlight nor analyse the risks inherent in subscribing to the equity issue of a VCT. [17] However - in recognition of these risks - the U.K. government provides investors

[17] VCTs are exposed to significantly higher risks than non-VCT equities. VCTs invest in smaller, fledgling firms, a lot of which will struggle or go into liquidation, resulting in losses for investors. Additionally, VCT shares are

with a 30% income tax relief for subscriptions in new VCT fundraising. To illustrate with an illustration drawn from HMRC (2018) venture capital trust statistics, assume an investor invests £10,000 in a VCT fundraising round. This investor either receives a £3,000 cheque from the tax authority or a £3,000 reduction in her tax bill. We should emphasise that this is a tax rebate, hence restricted to the amount of income tax she paid. This means that (and given that the maximum annual VCT investment is £200,000) if she has only paid £2,000 in income tax, she would only receive a £2,000 instead of £3,000 tax rebate on her £10,000 investment. She must also hold her VCT shares for five years to permanently keep the tax rebate. Also, she does not get the rebate if she bought the shares on the secondary market. This example also illustrates the fact that the tax benefits from VCT investments are dependent on each individual investor's circumstances. We further illustrate with three more examples:

Example A

Francesca decides to invest £200,000 in a VCT offer for subscription. In the 2019/20 tax year she anticipates that she will pay £90,000 in income tax.

Investment	£200,000
Tax Rebate	(£60,000)
Effective Net Cost	£140,000
Tax Rebate as a percentage	30%

Example B

In the tax year 2019/20, Bukola decides to invest £10,000 in a VCT offer for subscription. She is a basic rate and non-Scottish tax-payer; she earns £30,000 annually hence will pay approximately £3,500 in income tax $([30,000 - 12,500(\text{Personal Allowance})] \times 20\%)$.

Example C

Adesua wants to invest £100,000 in a VCT offer for subscription. She is a higher rate and non-Scottish tax-payer; she earns £60,000 annually and has calculated that she will pay £11,500[18]

illiquid. Even though their shares are fully listed on the London Stock Exchange, they might be difficult to sell, due to in some cases there being a single "market maker" for the shares.

[18]Her tax liability is calculated as the sum of 0% on £12,500 personal allowance, basic rate of 20% on £37,500,

Investment	£10,000
Tax Rebate	(£3,000)
Effective Net Cost	£7,000
Tax Rebate as a percentage	30%

in income tax in the tax year 2019/20.

Investment	£100,000
Tax Rebate	(£11,500)
Effective Net Cost	£88,500
Tax Rebate as percentage	11.5%

Adesua will not pay enough income tax to reclaim the full 30% tax rebate, hence will only receive the £11,500 in tax she paid as rebate.

D Major VCT Policy Changes

- 2004-06: 6th April 2004 - Introduction of the 40% income tax relief rate for a two-year period starting on 6 April 2004 - prior to which income tax relief was given at 20%.
  Also, from 6th April 2004, the maximum amount individual investors could invest in VCTs to qualify for income tax relief increased from £100,000 to £200,000.
  However, the holding period - to keep your income tax relief - for VCT shares held by investors increased from three to five years.
  We attribute the highest points (2004-2005) in our Fig.7 of aggregate annual investment to the increased income tax relief. Our assertion is backed by the 244% average increase in the amount of funds raised in both 2004 and 2005 relative to the average raised in the two years prior (See Table 2). In the aggregate, VCTs attributed the high levels of funds raised and the subsequent high level of investment to the increased income tax relief.

and higher rate of 40% on £10,000

- 2006-07: 6th April 2006 - The maximum gross assets of qualifying investees was reduced from £15 million to £7 million before investment and from £16 million to £8m immediately after investment. Also, the rate of income tax relief was reduced to 30% from 40%.

- 2007-08: 6th April 2007 - VCT qualifying investees must be firms with fewer than 50 full-time employees at the time shares are issued. 19th July 2007 - Investees can only raise a maximum of £2 million in any 12 month period under any or all of the tax-based venture capital schemes (Venture Capital Trusts, Enterprise Investment Scheme).
  Again, our analysis of the annual reports of each VCT managing funds within the 2006-2008 period reveals that the reduction in the rate of income tax relief - from 40% to 30% depressed their fundraising activities within the period. Most importantly - as explicitly reported by VCT investment managers in their annual reports - the reduction in the size of qualifying investees increased the risk profile of potential investees and further depressed their investment activities. All of this largely [19] ***explains the sustained downward trend in investment between 2006 - 2009 as seen in Fig.7. Our explanation also bears out in the numbers in Table 2. We see that the number of VCTs raising funds as a proportion of those managing funds drops from 68% in 2004-2006 to 34% in 2006-2008***

- 2009-10: Capital raised by VCTs in a share issuance should be fully employed within two years of the issuance. However, if the issue takes place before commencement of the intended trade, then the capital raised should be fully employed within two years of commencement.
  Our analysis reveals that the 2009-10 major policy change did not drive the upward trend seen in the same period, see Fig.7. During the period, VCT investment managers documented their concerns about the impact of the economic downturn and tightened lending conditions on SMEs. They however saw this as an opportunity to further invest in their existing portfolios; tightened lending conditions meant VCTs were one of the few sources of financing for investees: through the provision of working capital to investees, by funding acquisitions carried out by investees, and funding the restructuring of investees.

[19] We hedge by using the adverb "largely" because this time period also coincides with the height of the financial crisis and the attendant bear market. This however is an area we will not explore in this study.

Thus, we see from Table 2, that even though fundraising in the period was at a three-year high, the number of new investees that received VCT funding was the lowest it had been since 2003 (see Fig.2). This means, more money was being raised by VCTs relative to the last three years, but fewer new investees were receiving said funds. Therefore, the data in Fig.2 backs up the documented claim that VCTs viewed the tightened lending conditions for SMEs as an opportunity to solidify their existing positions under favourable terms, and hence, a large proportion of the three-year-record-breaking newly raised funds went to existing investees.

- 2010 - 2011: 6th April 2011 - VCTs must hold at least 70%, by VCT tax value, of its total investments (shares, securities and liquidity) in VCT qualifying holdings, within approximately three years of a fundraising. For VCTs whose accounting periods begin on or after 1 January 2020, this percentage increased to 80%. From that date, total investments also includes funds raised up to 31 December 2017.
  Also, a VCT can only invest a maximum of £1m per tax year in each of its investees, and no investment in a single investee or group of investees may constitute more than 15% (by VCT tax value) of the VCT's total investments at the date of investment.

- 2011-12: For funds raised before April 2011: at least 30% of a VCT's qualifying investment by value must be held in "eligible shares" (do not carry any preferential rights). For qualifying investments made by VCTs after 5 April 2018, together with qualifying investments made by funds raised after 5 April 2011, they must in aggregate be comprised of at least 70% by VCT tax value in "eligible shares".
  At least 10% of each investment in qualifying investees is held in eligible shares (by cost at the time of investment).
  A VCT's income must come wholly or primarily from shares and securities.
  VCTs must distribute sufficient dividends from their revenue available for distribution so as not to retain more than 15% of their income from shares and securities in a year.
  VCTs must be listed on a U.K. recognised Stock Exchange.
  The requirement that a potential investee's main trade be carried on wholly or mainly in the U.K. was cancelled, and replaced with a requirement that the investee have a permanent establishment in the U.K.

The restriction that prevented VCTs from investing more than £1m per annum in any single investee was also removed.

- 2012-13: 6th April 2012: The 2007 restriction on VCT qualifying investees having a maximum of 50 employees is increased to a maximum of 250 full time equivalent employees. Also, the 2006 reduction in gross assets of VCT qualifying investees was reversed. VCTs can once again invest in firms with maximum gross assets of £15 million before investment and £16 million after investment.
  Additionally, the rule that an investee is restricted to an annual VCT investment limit of £2m - imposed in 2007 - is increased to £5 million, with a lifetime limit of £12 million (for knowledge intensive companies the annual limit is £10 million and the lifetime limit is £20 million).
  Regarding investments made by a VCT from capital it raised on or after 6 April 2012, if an investee uses the funds to acquire shares in another company, this will not be considered as using them for a qualifying purpose.
  The main theme of the policy changes between 2010-2013 was a reversal of the 2006-2007 changes. These reversals were introduced to stimulate VCT fundraising and subsequent investment in U.K. SMEs.
  However, all of the investment managers expressed concern in their annual reports about an uncertain and fragile U.K. economy. The main highlights of their concern were the sovereign debt crisis in the eurozone, upward inflationary pressures, and a sustained downward pressure on public sector spending.
  These reasons help explain the downward trend we see in the period in Fig.7.

- 2014-15: From April 2014 VCTs could no longer return share capital to investors within three years of the end of the accounting period in which the VCT issued the shares.
  Additionally, legislation was introduced to prevent investors refreshing income tax relief on investments into VCTs by disposing of VCT shares and reinvesting the proceeds in new shares. The legislation allowed new investment into VCTs to still be eligible for income tax relief. However, investments that were:

  – conditional on a share buy-back or made within a six month period of a sale of shares in the same VCT would not qualify for income tax relief. The measure did not affect subscriptions for shares where the monies being subscribed represented dividends which the investor had elected to reinvest. The legislation was also changed to allow individuals to subscribe for shares in a VCT via a nominee.

These major policy changes are responsible for the downward trend depicted in the period in Fig.7.

- 2015-16: 8th July 2015 - Policy changes were introduced to bring the VCT scheme in line with the European Union's risk capital guidelines:

    1. VCTs may not: offer secured loans to investees, and any returns on loan capital above 10% must only represent a commercial return on the principal; invest in investees that do not meet the new "risk to capital" condition (which requires an investee, at the time of investment, to be an entrepreneurial company with the objective to grow and develop, and where there is a genuine risk of loss of capital).

    2. Restrictions on investments that VCTs can make, particularly with respect to the age of the business. Potential investees have been limited to firms that are less than 7 years old (ten years for knowledge intensive businesses).

       Non-qualifying investments can no longer be made, except for certain exemptions in managing the Company's short-term liquidity. Exemptions are limited to investments in firms such as OEICs (Open Ended Investment Company), Investment Trusts or listed firms.

       Investment managers report that this policy change (No.2) will curtail their investment in Alternative Investment Market (AIM) shares; AIM shares form a significant proportion of VCT portfolio holdings. This line of reasoning is clearer when we consider that the London Stock Exchange requires that firms be at least 3 years old before they can registered on the AIM.

       VCTs further interpret this particular policy change as likely to reduce the scope of investments they can make, potentially increasing the risk profile of their portfolios. For instance, they claim that replacing the shares of AIM firms with that of smaller unquoted firms will increase the risk profile of their portfolios.

3. Ban on using funds raised by VCTs to finance Management Buyout (MBO), Buy-In Management Buyout (BIMBO), or company acquisitions. ***Investment managers report that this will eliminate the lower risk component of their portfolios***.
4. These policy changes were introduced with a ten-year sunset clause - providing a decade of stability with regards VCT policy changes.

In summary, we have two countervailing forces affecting VCTs. On the one hand, the narrower set of investment opportunities (No's 2, 3, 4) could potentially depress investment activity. To paraphrase the sentiments of numerous investment managers "These new inhibitions will curtail significant drivers of growth in the U.K. SME ecosystem. They will curtail, as opposed to encourage, investment activity". On the other hand - and this sentiment was also explicitly expressed by VCT investment managers in their annual report - there is a high demand for VCTs to fundraise as a result of a reduction in the pension lifetime allowance from £1,250,000 to £1,000,000, the tapering away of pension tax allowances for high earners earning £110,000 a year or more, which can gradually reduce said person's annual allowance from the standard £40,000 to as low as £10,000[20], and the launch of pension freedoms that allow for cash to be taken out of the pot for investment rather than buying an annuity. All of this has caused VCTs to become more attractive to investors seeking additional tax-advantaged investments.

The tax-advantage phenomena clearly dominated the narrower set of investment opportunities phenomena, and helps explain the upward trend we see in investment beginning in 2015 till the end of our sample in 2018.

What is clear from the major VCT policy changes between 2015-18 is the Government's desire to refocus investment towards young growth companies. We have argued that these changes have been successful in stimulating new investment, especially the ban on funding MBOs, BIMBOs and acquisitions. To reiterate the point, our reader might have noticed that prior to 2015, periods of rising growth in the rate of investment always preceded or followed periods of falling growth in investment. However, since 2015, the growth in investment has been trending upward.

[20]Prior to 2009, high earners could save up to £235,000 a year in a pension and receive nearly £100,000 in tax relief. As of 6th April 2016, that sum is limited to £10,000 in a pension and just £4,000 in tax relief.

- 2017-2018: Patient Capital Review:

  In the November 2017 budget, the U.K. Government reviewed the VCT scheme as part of its wider "Patient Capital Review"[21]. The outcome was a number of proposed changes to the VCT regulations in an effort to refocus investment on potentially higher risk sectors that require capital (Her Majesty's Treasury Policy Paper, 2017) - summarised below:

  1. Expand the VCT scheme to enable VCTs to provide follow-on investment which will help to "scale-up" investees, thus easing the transition from a dependence on VCT funding to venture funding. For instance, increasing the current Knowledge Intensive Company allowance would help increase the focus on science based firms.
  2. Increasing the annual and lifetime investment limits would allow for follow on investment from VCTs, thus slowing the transition away from tax-incentivised financing (Her Majesty's Treasury Policy Paper, 2017).

  Anticipation of the above changes from the Patient Capital Review also influenced the increased growth rate in investment between 2017-2018.

- April 2020: Minimum of 80% of a VCT's funds must be invested in VCT qualifying investments - up from 70%.

[21] The review considered how to support innovative firms to access the finance that they need to scale up. Her Majesty's Treasury published a consultation seeking views on how to increase the supply of capital to growing, innovative firms.

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